Rational Stock-Market Fluctuations

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1

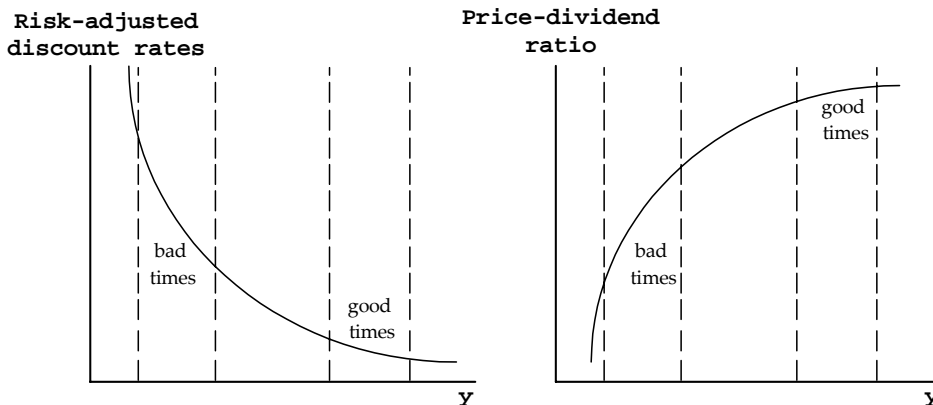# Rational Stock-Market Fluctuations

Abstract

This paper introduces a theory illustrating how rational models may help to explain the central features of aggregate stock-market fluctuations: the systematic occurrence of procyclical movements in price-dividend ratios, the countercyclical variation of expected returns and returns volatility, and the positive relation linking expected dividend growth to expected returns. I provide a series of joint restrictions on variables affecting interest rates, dividend growth and risk-premia leading to the previous properties within a fairly general class of dynamic economies. These restrictions shed new light on the key mechanism by which primitive assumptions exert their influence on asset prices, and offer practical guidance on model design.

Understanding the properties of aggregate stock-market behavior has long been the subject of both theoretical and empirical research in financial economics. While the statistical properties of the aggregate stock-market now seem to be well-understood, there still is a large variety of theoretical models which compete to rationalize the empirical findings. Perhaps surprisingly, the general properties of these theoretical models are poorly understood. As an example, we do not have a theory able to answer such questions as: When are price-dividend ratios procyclical? When is stock-market volatility countercyclical? When are expected returns positively related to expected dividend growth? This paper introduces a theory which explicitly addresses these and related questions.

The perspective taken in this article differs from previous approaches in one fundamental respect. Typically, asset pricing predictions rely upon assumptions related to both the intertemporal marginal rate of substitution, or more generally the pricing kernel (i.e. interest rates and risk-premia), *and* dividends distributions. Alternatively, this article looks for: 1) pricing kernels consistent with given dynamic properties of asset prices *and* dividends; and 2) properties of dividends consistent with given dynamic properties of asset prices *and* pricing kernels. All in all, the theory in this article provides restrictions on primitives (variables affecting pricing kernels, and dividends) which make the resulting asset price process consistent with a variety of patterns of aggregate stock-market behavior given in advance. In turn, this approach leads to a rich array of new joint restrictions on the dynamics of price-dividend ratios, expected returns, returns volatility and expected dividend growth. Finally, a very important feature of these restrictions is that they apply to a rich class of dynamic economies. For this reason, they offer many new insights into the key general economic mechanisms through which fundamentals move asset prices.

The restrictions developed in this article can not be anticipated by *intuitive* reverse engineering. For example, it is relatively easy to design models in which risk-premia are countercyclical. Nevertheless, countercyclical risk-premia do not imply countercyclical returns volatility. This paper shows that to understand such an additional important issue, it is very important to foresee the exact amount of risk-premia curvature with respect to variables tracking the state of the economy (see Figure 1 below and the ensuing discussion). Typically, standard intuition provides insights into the asset pricing implications of "first-order" properties of risk-premia. For example, economic intuition may help to pin down models in which risk-premia are decreasing in some state variables correlated with the state of the economy. The restrictions in this paper also yield precise insights into the asset pricing implications of *higher order* properties (e.g., curvature) of risk-premia. Moreover, these restrictions accomplish this task in a systematic and constructive way - that is, they tell a model designer which "parameters" are responsible of a given property.

3

Figure 1 – Countercyclical returns volatility

If price-dividend ratios are concave in some state variable y tracking the business cycle conditions, returns volatility increases on the downside, and is consequently countercyclical. According to the theory in this article, price-dividend ratios are concave in y whenever (risk-adjusted) discount rates are decreasing and *sufficiently* convex in y.

The main points of this paper can be illustrated through two examples. First, consider the standard models of aggregate stock-market fluctuations with external habit formation. These models may predict countercyclical returns volatility. For instance, Campbell and Cochrane (1999) found this property by relying on the numerical solution of their model. According to the theory in this article, returns volatility is countercyclical if the pricing kernel in the economy is such that risk-adjusted discount rates are decreasing and *sufficiently* convex in some state variable tracking the business cycle conditions (y say). This convexity result is in fact very general, that is it applies to economies beyond the example of habit formation (see proposition 2 in section 3.2). And its economic interpretation is quite simple: Due to convexity, in good times investors do not change too much the discount factor they use to evaluate future dividends (see Figure 1); as a consequence, price-dividend ratios do not fluctuate too widely. On the other hand, in bad times investors change in an extremely sensitive way their discount factors in response to changes in the economic conditions; as a result, variations in the price-dividend ratio become increasingly volatile as business-cycle conditions deteriorate. [1] If such an asymmetry in discounting is sufficiently strong, price-dividend ratios are increasing and concave in y, and volatility increases on the downside. More precisely, in proposition 2 I calculate the amount

4

of asymmetry in discounting activating the previous effects. This amount of asymmetry can be very stringent: there can be models predicting countercyclical risk-premia that at the same time do not satisfy the conditions in proposition 2 - and consequently induce price-dividend ratios to be convex in the state variables tracking the business cycle conditions (see Table 1 in section 3.2). Naturally, countercyclical returns volatility may also emerge simply because the state variables in the economy display countercyclical volatility in the first place. Alternatively, the conditions developed here highlight the mechanism through which countercyclical returns volatility is *endogenously* induced by rational fluctuations in the price-dividend ratios.

The learning models introduced by Veronesi (1999) constitute a second example the theory of this article applies to. These models predict that asset prices are increasing and convex in the agents' posterior probability of the economy being in a good state. Veronesi offered many insights on such a *rational* "excess" price-sensitivity. The theoretical test conditions of this article provide further precise insights on this and other learning models [such as Brennan and Xia (2001) model]. They point to two main conclusions. First, the overreaction property observed by Veronesi is a robust property shared by other learning models. Second, the same property is the manifestation of a more general characteristic of any long-lived asset pricing model. More precisely, I find that the price of a long-lived security has the intriguing property to inherit the same features of the expected changes in dividend growth. Intuitively, the pricing function of a security is a risk-adjusted (discounted) expectation of the future dividends stream. Therefore, the (risk-neutral) expected changes in dividend growth do play a fundamental role. In particular, I find that in continuous time, the pricing function is convex in the expected dividend growth whenever the (risk-adjusted) drift of dividend growth is convex - or at most, linear (see proposition 1 in section 3.1). Put differently, expected dividend growth should fluctuate according to unusual patterns to generate non-convex pricing functions.

Not only does the previous convexity property help to better understand rational overreaction. Under additional technical conditions, this property may also induce a positive relationship between expected returns and expected dividend growth - a very important feature of data documented by Lettau and Ludvigson (2003). Intuitively, this relationship emerges according to the following mechanism. In the economies I consider here, risk-aversion makes expected returns *positively* related to volatility of price-dividend ratios. And by a reasoning similar to the one leading to the *concavity* effects in Figure 1, this volatility now increases with the expected dividend growth if price-dividend ratios are increasing and *convex* in the expected dividend growth. At the same time, I emphasize that all these convexity results may be weakened by excessive interest rates variability (see section 3.1 for a comprehensive discussion).

5

The previous two predictions of the theory (about habit models and learning/dividend growth models) are part of a more elaborate, multidimensional framework of analysis. This framework encompasses two categories of multidimensional models, each having its own economic motivation. Both categories extend the standard Lucas (1978) model of the (single) Markov consumption good process. The extensions operate along the two most natural dimensions. In the first one, one state variable affects expected dividend growth (as for example in the learning models mentioned above) (see section 3.1). In the second one, one state variable affects risk-adjusted discount rates (as for example in the habit models mentioned above) (see section 3.2). Multidimensional extensions of these models are considered in the more technical section 4 (see proposition 3 in section 4). In all these settings, agents have fully rational expectations. The only additional assumptions that I make are that the state variables of the economy are Markov processes with continuous sample paths (i.e. diffusion processes) satisfying some basic regularity conditions, and that asset prices are arbitrage-free and homogeneous in the dividend process.

While the restrictions developed in this article apply to fairly general economies, one attractive feature of them is that they are simple to check. For this reason, they can be used to understand intricate properties of models directly from primitive assumptions - i.e. without resorting to approximations, simplifying assumptions, or numerical methods. [2] The final contribution of this paper is then to show how to use these restrictions to provide a general (and simple) unifying characterization of many existing models: precisely, I reconsider a number of models in the literature (as well as some new ones), and I use the theory in this article to gain several novel insights into the critical properties of these models. At the same time, these examples help to further illustrate the key points of the theory.

Finally, the theory in this article is related to the "integrability" problem studied by He and Leland (1993), Wang (1993), Dybvig and Rogers (1997), Cuoco and Zapatero (2000), and others. [See, also, Ang and Liu (2004) for a related approach.] The integrability problem consists in recovering preferences (and beliefs) from the knowledge of a given equilibrium asset price process. Here, I also derive restrictions which make pricing kernels consistent with a given rational asset price process. One distinctive feature of this article is that it is not confined to settings with complete markets and/or standard additive expected utility functions. In this respect, the approach in this paper is very close in spirit to the Constantinides and Duffie (1996) search for income processes that are compatible with a given aggregate income and asset price process. Furthermore, I consider multidimensional settings and I provide accurate descriptions of *both* "implied" kernel *and* dividends processes. On the other hand, the theoretical test conditions developed here impose restrictions on kernels and other primitives of *reduced-form* economies.

The article is organized in the following manner. The next section describes the primitives of the model. Section 2 outlines how the motivational issues of this introduction are addressed in this paper. Section 3 is the main core of the paper. It examines models including learning mechanisms, stochastic dividend growth, and time-varying risk-aversion. Section 4 extends the theory to four-factor models. Section 5 concludes. Three appendices gather technical details.

1   The asset pricing model

I consider a pure exchange economy endowed with a flow of a (single) consumption good. Let $\{D_t\}_{t\geq0}$ be the process of instantaneous rate of consumption endowment. With the exception of section 4, I assume that consumption equals the dividends paid by a long-lived asset. Accordingly, the terms "consumption" and "dividends" are used interchangeably. Let $\{y_t\}_{t\geq0}$ be an additional state variable. I assume that (D_t, Y_t) form a diffusion process, with $D_0 = D$ and $y_0 = y$ (say), where $(D, y) \in \mathbb{D} \times \mathbb{Y}$, $\mathbb{D} \subset \mathbb{R}_{++}$ and $\mathbb{Y} \subset \mathbb{R}$. Consequently, I fix a probability space (Ω, F, Pr) and a family $\{F_t\}_{t\geq0}$ of sigma-algebras that is the augmented filtration of a standard Brownian motion in $\mathbb{R}^2$. As shown in section 3, this simple setting is general enough to include many existing models. Extensions to higher dimensions are considered in section 4.

A long-lived asset is an asset that promises to pay $\{D_t\}_{t\geq0}$. Let $\{P_t\}_{t\geq0}$ be the corresponding asset price process. As is well-known, absence of arbitrage opportunities implies that there exists a positive pricing kernel $\{\xi_t\}_{t\geq0}$ such that

$$P_t \xi_t = E\left[\int_t^\infty \xi_s D_s ds\right], \quad t \geq 0, \tag{1}$$

where E is the expectation operator taken under probability measure Pr. "Bubbles" are not considered in this paper. Moreover, I only consider classes of models predicting that the price *process* P_t satisfies the Markov property: $P_t \equiv P(D_t, y_t)$, where *function* $P(D, y) \in \mathcal{C}^{2,2}(\mathbb{D} \times \mathbb{Y})$ (the space of continuous and twice continuously differentiable functions on $\mathbb{D} \times \mathbb{Y}$). A simple condition ensuring the existence of such a price function P is that (D_t, y_t, ξ_t) satisfy:

$$\begin{cases} dD_t = G(D_t, y_t)\,dt + \sigma(D_t)\,dW_{1t} \\[2mm] d\xi_t = -\xi_t\left[R(D_t, y_t)\,dt + \lambda_1(D_t, y_t)\,dW_{1t} + \lambda_2(D_t, y_t)\,dW_{2t}\right] \\[2mm] dy_t = m(D_t, y_t)dt + v_1(y_t)\,dW_{1t} + v_2(y_t)\,dW_{2t} \end{cases} \tag{2}$$

where W_1 and W_2 are independent standard Brownian motions; G, σ, m, v_1 and v_2 ($\sigma > 0$, $v_i > 0$, $i = 1, 2$) are given functions guaranteeing a strong solution to the previous system; and R, λ_1 and

λ_2 are functions satisfying all regularity conditions needed for the representation in eq. (1) to exist. As is also well-known, R is the instantaneous (or short-term) rate process, and $\lambda \equiv [\lambda_1 \ \lambda_2]^{\top}$ is the vector of unit prices of risk associated with the sources of risk W_1 and W_2. Finally, it is also well-understood that the price P can be expressed as,

$$P(D, y) = \int_0^{\infty} C(D, y, t)dt, \quad C(D, y, t) \equiv \mathbb{E}\left[\exp\left(-\int_0^t R(D_u, y_u)dt\right) \cdot D_t \,\Big|\, D, y\right], \quad (3)$$

where $\mathbb{E}$ is the expectation operator taken under the risk-neutral probability Q (say). Under this probability measure, (D_t, y_t) are solution to

$$\begin{cases} dD_t = \hat{G}\left(D_t, y_t\right)dt + \sigma\left(D_t\right)d\hat{W}_{1t} \\ \\ dy_t = \hat{m}(D_t, y_t)dt + v_1\left(y_t\right)d\hat{W}_{1t} + v_2\left(y_t\right)d\hat{W}_{2t} \end{cases} \quad (4)$$

where $\hat{W}_1$ and $\hat{W}_2$ are two independent Q-Brownian motions, and $\hat{G}$ and $\hat{m}$ are risk-adjusted drift functions defined as $\hat{G}\left(D, y\right) \equiv G\left(D, y\right) - \sigma\left(D\right)\lambda_1\left(D, y\right)$ and $\hat{m}\left(D, y\right) \equiv m\left(D, y\right) - v_1\left(y\right)\lambda_1\left(D, y\right) - v_2\left(y\right)\lambda_2\left(D, y\right)$. [3]

The objective of the article is to develop general properties of the rational pricing mapping $(D, y) \mapsto P(D, y)$ under the additional technical condition that P and its partial derivatives may be represented through the Feynman-Kac theorem. [4] I now turn to illustrate the main issues motivating such a level of analysis.

2 Issues

This article singles out general properties of long-lived asset prices that can be streamlined into two categories: "monotonicity properties" and "convexity properties". I now illustrate the economic content of such a categorization. I first formulate a crucial assumption which will considerably simplify the presentation:

Assumption 1. (Scale-invariant economies) The asset price function $P(D, y) = D \cdot p(y)$, for some positive function $p \in \mathcal{C}^2(\mathbb{Y})$.

The previous assumption is not satisfied in representative agent models with non-separable utility over several goods [as, e.g., in Aït-Sahalia, Parker and Yogo (2004)] and/or general HARA utility function. It is thus a very restrictive assumption. Its main advantage is merely expositional. Scale-invariant economies still arise in a number of models capable to explain important

characteristics of asset prices and returns (see, e.g., examples 1 to 4 below); assumption 1 helps to isolate the key properties of these models in a relatively simple way. The theory in this article is much more cumbersome to illustrate in classes of models not satisfying assumption 1. Nevertheless, economic intuition gained with models satisfying assumption 1 carries over more general models. It is emphasized that the *general* proofs in the appendix do not rely on assumption 1.

Two sets of primitive conditions on eqs. (2) ensuring that assumption 1 holds are: *a*) dividend volatility σ, expected dividend growth G, and the drift m of the state variable y satisfy: $\sigma(D) = \sigma_0 D$ for some constant $\sigma_0 > 0$, $G(D, y) = g(y) D$ for some function g, and $m(y) \equiv m(D, y)$; and *b*) the short-term rate R and the unit-risk premia λ_i are functions of the state variable y only, viz $R(y) \equiv R(D, y)$ and $\lambda_i(y) \equiv \lambda_i(D, y)$ $(i = 1, 2)$ (see appendix A for the proof of this claim). All model examples in this paper do satisfy these conditions. By these conditions, and the standard pricing equation $0 = \xi D dt + E[d(\xi P)]$ (see, e.g., Cochrane, 2001), a simple application of Itô's lemma to the definition of returns ($\frac{dP + D dt}{P}$) leaves,

$$\text{Returns}_t = [R(y_t) + \mathcal{E}(y_t)] dt + \left[\sigma_0 + \frac{p'(y_t)}{p(y_t)} v_1(y_t)\right] dW_{1t} + \frac{p'(y_t)}{p(y_t)} v_2(y_t) dW_{2t},$$

where

$$\begin{cases} \text{Expected Excess Returns} & \equiv \mathcal{E}(y) = \mathcal{V}(y) \cdot \sigma_\xi(y) \cdot [-corr(\text{Returns}, \ d\xi/\xi)] \\[4mm] \text{Returns Volatility} & \equiv \mathcal{V}(y) = \sqrt{\left[\sigma_0 + \frac{p'(y)}{p(y)} v_1(y)\right]^2 + \left[\frac{p'(y)}{p(y)} v_2(y)\right]^2} \end{cases} \tag{5}$$

and $\sigma_\xi \equiv \|\lambda\|_2$, the volatility of the pricing kernel ξ in eqs. (2).

The previous formulas make it clear why it is important to characterize properties of the price-dividend ratio p. If we are able to understand properties of p directly from first principles, we can derive straight forward implications for the dynamics of expected returns, volatility and Sharpe ratios (defined as $\mathcal{E}(y)/\mathcal{V}(y)$). Monotonicity and convexity properties of p are the most natural properties to investigate for a number of reasons including the following ones:

- *Monotonicity.* As is well-known, empirical evidence suggests that actual returns volatility is too high to be explained by consumption volatility [see, e.g., Campbell (2003) for a survey]. Naturally, additional state variables may increase the overall returns volatility. In the simple economy of this section, state variable y inflates returns volatility whenever the price-dividend ratio p is increasing in y. At the same time, such a monotonicity property would ensure that returns volatility be strictly positive - one crucial condition guaranteeing that dynamic constraints of optimizing agents are well-defined.

- *Convexity*: *I.* Next, suppose that y is some state variable related to the business cycle conditions. Another robust stylized fact is that stock-market volatility is countercyclical [see, e.g., Schwert (1989)]. In the class of economies considered in this section, returns volatility is countercyclical whenever p is increasing and *concave* in y, and the two sources of volatility of the state variable y (v_1 and v_2) are non-increasing. Even in this simple example, second-order properties (or "nonlinearities") of the price-dividend ratio p are critical to understanding time variation in returns volatility.

- *Convexity*: *II.* Alternatively, consider a model in which expected dividend growth is positively affected by a state variable y. If the price-dividend ratio p is increasing and *convex* in y, price-dividend ratios would typically display "overreaction" to small changes in y when y is high. The empirical relevance of this point was first recognized by Barsky and De Long (1990, 1993) [see also Timmermann (1993) for a related approach]. [5] More recently, Veronesi (1999) addressed similar convexity issues by means of a fully articulated equilibrium model of learning.

Motivated by the previous points, I now derive a characterization of the price-dividend ratio p. The general question I wish to answer is, Which joint restrictions on R, λ_i, m and v_i in eqs. (2) are needed to make the price-dividend ratio monotone, concave or convex in y? A key insight at this juncture is that according to eq. (3), a long-lived asset price $P(D,y)$ is a linear functional of European-type option prices $\{C(D,y,t)\}_{t\geq 0}$. Moreover, under assumption 1, the same phenomenon occurs for the price-dividend ratio,

$$p(y) = \int_0^\infty c(y,t)\,dt, \quad c(y,t) \equiv \mathbb{E}\left[\exp\left(-\int_0^t R(y_u)\,dt\right)\cdot \frac{D_t}{D}\bigg| y\right], \qquad (6)$$

$$\frac{D_t}{D} = \exp\left[\sigma_0 \hat{W}_{1t} + \int_0^t (g(y_u) - \sigma_0 \lambda(y_u))\,du\right].$$

The main idea now is to understand properties of the price-dividend ratio p through the corresponding properties of European-type contingent claim prices $\{c(y,t)\}_{t\geq 0}$ in eq. (6). We have:

Lemma 1. *Let $\{y_t\}_{t\geq 0}$ be the (strong) solution to*:

$$dy_t = b(y_t)\,dt + a(y_t)\,d\hat{W}_t,$$

where $\hat{W}$ is a multidimensional Q-Brownian motion and b, a are some given functions (a is vector-valued). Let ψ and ρ be two twice continuously differentiable positive functions, and define

$$c(y,T) \equiv \mathbb{E}\left[\exp\left(-\int_0^T \rho(y_t)\,dt\right)\cdot \psi(y_T)\bigg| y\right]. \qquad (7)$$

The following statements are true:

a) If $\psi' > 0$, then c is increasing in y whenever $\rho' \leq 0$. Furthermore, if $\psi' = 0$, then c is decreasing (resp. increasing) whenever $\rho' > 0$ (resp. < 0).

b) If $\psi'' \leq 0$ (resp. $\psi'' \geq 0$) and c is increasing (resp. decreasing) in y, then c is concave (resp. convex) in y whenever $b'' < 2\rho'$ (resp. $b'' > 2\rho'$) and $\rho'' \geq 0$ (resp. $\rho'' \leq 0$). Finally, if $b'' = 2\rho'$, c is concave (resp. convex) whenever $\psi'' < 0$ (resp. > 0) and $\rho'' \geq 0$ (resp. ≤ 0).

Lemma 1 introduces many new results and merits discussion. First, lemma 1-a) generalizes previous monotonicity results obtained by Bergman, Grundy and Wiener (1996). By the so-called "no-crossing property" of a diffusion, y_t is not decreasing in its initial condition y. Therefore, c inherits the same monotonicity features of ψ if discounting does not operate adversely. While this observation is relatively simple, it explicitly allows to address monotonicity properties of long-lived security prices.

Second, lemma 1-b) generalizes a number of existing results on contingent claims price convexity. Assume for example that ρ is constant and that y_t is the price of a traded asset. In this case, $\rho' = b'' = 0$. The last part of lemma 1-b) then says that convexity of ψ propagates to convexity of c. This result reproduces an important finding in the option pricing literature [Bergman, Grundy and Wiener (1996) and El Karoui, Jeanblanc-Picqué and Shreve (1998)]. Lemma 1-b) characterizes option price convexity within more general contingent claims models. Its most surprising implication is that interesting nonlinearities emerge in the presence of nontradable state variables. In fact, lemma 1-b) reveals that convexity of the terminal payoff ψ is neither a necessary nor a sufficient condition for convexity of c. As an example, suppose that $\psi'' = \rho' = 0$ and that y_t is not a traded risk. Then, lemma 1-b) reveals that c inherits the same convexity properties of the instantaneous drift of y_t. Finally, lemma 1-b) extends one (scalar) bond pricing result in Mele (2003). Precisely, let $\psi(y) = 1$ and $\rho(y) = y$; accordingly, c is the price of a zero-coupon bond as predicted by a standard short-term rate model. By lemma 1-b), c is convex in y whenever $b''(y) < 2$ for all y. This corresponds to eq. (8) (p. 688) in Mele (2003). In analyzing properties of long-lived security prices, both discounting and drift nonlinearities play a prominent role. For the purpose of this paper, I therefore need the more general statements contained in lemma 1-b).

3   Core theory

Models in which long-lived asset prices are driven by only one state variable fail to explain the actual characteristics of aggregate stock-market behavior. The simplest multidimensional exten-

sions are obtained through the introduction of 1) time-varying expected dividend growth and/or 2) time-varying risk-adjusted discount rates. In section 3.1, I develop properties of models addressing the first extension. Properties of models with time varying discount rates are investigated in section 3.2. In this section, I thus aim at disentangling the effects of random changes in average profitability from the effects of random changes in discount rates. This helps to develop intuition on the functioning of the more complex multifactor model in section 4.

3.1 Stochastic dividend growth

In this section, I consider economies in which the instantaneous rate of consumption endowment, or dividend process $\{D_t\}_{t \geq 0}$ satisfies

$$
\begin{cases}
\dfrac{dD_t}{D_t} = g\left(y_t\right) dt + \sigma_0 dW_{1t}, & \sigma_0 > 0 \\[2mm]
dy_t = m\left(y_t\right) dt + v_1\left(y_t\right) dW_{1t} + v_2\left(y_t\right) dW_{2t}, & v_1, v_2 > 0
\end{cases}
\tag{8}
$$

In this section, I also assume that the unit risk-premia in eqs. (2) $\lambda_i(D, y) = \lambda_i$ $(i = 1, 2)$, where λ_i are two constants. Therefore, the state variable $\{y_t\}_{t \geq 0}$ only affects the expected dividend growth in this model. For future reference, it is useful to introduce notation for the risk-neutralized drift function of state variable y_t, say $\hat{m}\left(y\right)$,

$$
\hat{m}\left(y\right) \equiv m\left(y\right) - \lambda_1 v_1\left(y\right) - \lambda_2 v_2\left(y\right).
\tag{9}
$$

I now provide examples of models that are special cases (or related) to the framework covered in this section.

Example 1. [Veronesi (1999, 2000)]. Consider an infinite horizon economy in which a representative agent observes realizations of D_t generated by:

$$
dD_t = \theta dt + \sigma_0 dw_{1t},
\tag{10}
$$

where w_1 is a Brownian motion, and θ is a two-states $(\bar{\theta}, \underline{\theta})$ Markov chain. θ is unobserved, and the agent implements a Bayesian procedure to learn whether she lives in the "good" state $\bar{\theta} > \underline{\theta}$. This economy is isomorphic in its pricing implications to one in which (D_t, y_t) are solution to:

$$
\begin{cases}
dD_t = y_t dt + \sigma_0 dW_{1t} \\[2mm]
dy_t = k\left(\bar{y} - y_t\right) dt + v_1\left(y_t\right) dW_{1t}
\end{cases}
$$

12

where W_1 is another Brownian motion, $v_1(y) = (\bar{\theta} - y)(y - \underline{\theta})/\sigma_0$, $k, \bar{y}$ are some positive constants. A related model which directly fits into model (8) is one in which D_t is solution to:

$$\frac{dD_t}{D_t} = \theta dt + \sigma_0 dw_{1t}, \tag{11}$$

and the agent receives additional signals $\{a_t\}_{t>0}$ about θ satisfying:

$$da_t = \theta dt + \sigma_1 dw_{2t},$$

where w_2 is a Brownian motion independent of w_1. Similarly as for model (10), the no-arbitrage price in this economy is isomorphic to the no-arbitrage price in an economy in which (D_t, y_t) are solution to eq. (8), with $g(y) = y$, $m(y) = \kappa(y^* - y)$, $v_1(y) = (\bar{\theta} - y)(y - \underline{\theta})/\sigma_0$, $v_2(y) = \frac{\sigma_0}{\sigma_1} v_1(y)$ and $f, \bar{g}$ are some positive constants. [6]

Example 2. A single infinitely lived agent observes D_t, where D_t is solution to,

$$\frac{dD_t}{D_t} = \theta_t dt + \sigma_0 dw_{1t}.$$

Similarly as in example 1, $\{\theta_t\}_{t>0}$ is unobserved. Unlike example 1, $\{\theta_t\}_{t\geq0}$ does not evolve on a countable number of states. Rather, it follows an Ornstein-Uhlenbeck process:

$$d\theta_t = k(\bar{g} - \theta_t)dt + \sigma_1 dw_{1t} + \sigma_2 dw_{2t},$$

for some positive constants $\bar{g}$, σ_1 and σ_2. The agent implements a learning procedure similar as in example 1. If she has a Gaussian prior on θ_0, the no-arbitrage price takes the form $P(D, y)$, where (D_t, y_t) now solve eqs. (8), with $g(y) = y$, $m(y) = k(\bar{g} - y)$, v_1 is constant, and $v_2 = 0$. [7]

The models in the previous examples share the same basic economic motivation. Yet they make different assumptions on the probabilistic structure of the unobserved consumption endowment growth rate. Do these assumptions entail different asset pricing implications? More generally, which minimal assumptions must *any* two "stochastic consumption growth" models share to display comparable pricing properties? Clearly, examples 1 and 2 only contain two possible kinds of models with incomplete information and learning mechanisms. [8] Furthermore, models making expected consumption another *observed* state variable may have an interest on their own [see, e.g., Campbell (2003) and Bansal and Yaron (2004)]. In this case, there might be no practical guidance as to how to choose a dynamic model of expected consumption changes.

The following proposition allows one to gauge the implications of primitive assumptions on the form of the asset price function.

Proposition 1. *Let the dividend endowment be as in eqs. (8), let interest rates be independent of dividends (i.e., $R(D, y) = R(y)$) and, finally, let λ_i ($i = 1, 2$) be constant. Then, the price function $P(D, y) = D \cdot p(y)$ for some positive function p with the following properties:*

a) p is increasing (resp. decreasing) whenever $R'(y) < g'(y)$ (resp. $R'(y) > g'(y)$) for all $y \in \mathbb{Y}$.

b) Let p be increasing; then p is convex (resp. concave) whenever $g''(y) - R''(y) > 0$ (resp. < 0) and $\hat{m}''(y) + 2[g'(y) - R'(y)] + \sigma_0 v_1''(y) > 0$ (resp. < 0) for all $y \in \mathbb{Y}$.

It is instructive to go through some steps to prove this proposition. Under the assumptions in this section, the price-dividend ratio p in eq. (6) satisfies:

$$p(y) = \int_0^\infty B(y, t)dt, \tag{12}$$

where

$$B(y, t) \equiv \mathbb{E}\left[\beta_t \cdot \exp\left(-\int_0^t (R(y_u) - g(y_u) + \sigma_0\lambda)du\right)\bigg|y\right] = \bar{\mathbb{E}}\left[\exp\left(-\int_0^t (R(y_u) - g(y_u) + \sigma_0\lambda)du\right)\bigg|y\right]. \tag{13}$$

and $\bar{\mathbb{E}}$ is the expectation under a new probability measure $\bar{Q}$ defined as $d\bar{Q}/dQ = \beta_t \equiv \exp(-\frac{1}{2}\sigma_0^2 t + \sigma_0\hat{W}_{1t})$, where $\hat{W}_1$ is the Brownian motion under the risk-neutral probability measure Q in eqs. (4). Intuitively, such a change of measure arises because D_t and y_t are correlated. Accordingly, y_t is solution to:

$$dy_t = [\hat{m}(y_t) + \sigma_0 v_1(y_t)]dt + v_1(y_t)d\bar{W}_{1t} + v_2(y_t)d\bar{W}_{2t},$$

where $\bar{W}_{1t} = \hat{W}_{1t} - \sigma_0 t$ is a $\bar{Q}$-Brownian motion, $\bar{W}_2 = \hat{W}_2$. By lemma 1, we can characterize p through the properties of B in eq. (13). To cast this problem in the format of lemma 1, set $\rho(y) \equiv R(y) - g(y) + \sigma_0\lambda$ and $b(y) \equiv \hat{m}(y) + \sigma_0 v_1(y)$. Monotonicity properties ($p' > 0$) now follow by the "no-crossing" property of a diffusion. Convexity properties follow by lemma 1-b).

Discussion

Proposition 1 imposes joint restrictions on the law of motion of the state process y_t (m and v_i) and levels of risk-aversion (λ_i). These restrictions then translate into predictions on: a) the

14

joint dynamics of expected returns, returns volatility and changes in y; and b) "overreaction" (or convexity) of price-dividend ratios. Consider for example the effects on expected returns. By the same calculations leading to eqs. (5) in section 2, one may easily show that in the class of economies considered in this section, expected (non-percentage) excess returns are proportional to,

$$\left[\sigma_0 p\left(y\right) + v_1\left(y\right) p'\left(y\right)\right] \lambda_1 + v_2\left(y\right) p'\left(y\right) \lambda_2, \quad \lambda_1, \lambda_2 \text{ constants.}$$

Next, suppose that the two sources of expected growth volatility v_1 and v_2 are non-decreasing - e.g., constant (just as in example 2), or increasing for a significant portion of variation of the state variable y affecting dividend growth (just as in example 1). If the unit risk premia λ_1 and λ_2 are positive, and price-dividend ratios $p\left(y\right)$ are convex in y, time-varying expected dividend growth may then induce a *positive* relation between expected returns and price-dividend ratios. Menzly, Santos and Veronesi (2004) have recently demonstrated that should such a property occur in multidimensional settings, price-dividend ratios would then be *weak* predictors of future dividend growth - a well-known empirical feature [see, e.g., Campbell and Shiller (1988)]. The theory in this section isolates precise conditions under which price-dividend ratios are convex. In section 4, I develop its multidimensional extensions in which the unit risk-premia λ_i may be driven by additional state variables. [9]

Would we generally expect that price-dividend ratios are convex in the expected dividend growth? To address this issue, it is instructive to apply the predictions of proposition 1 to the models in examples 1 and 2 - in which expected dividend growth g is simply $g\left(y\right) = y$. According to proposition 1, price-dividend ratios are *increasing* in the expected dividend growth y whenever the derivative $R'\left(y\right) < 1$; and *convex* in y whenever

$$\text{For all } y \in \mathbb{Y}, \quad \left(\sigma_0 - \lambda_1\right) v_1''\left(y\right) - \lambda_2 v_2''\left(y\right) - 2R'\left(y\right) > -2. \tag{14}$$

Consider then eq. (11) in example 1 and assume that in this economy, a) markets are complete; and b) there is a representative agent with CRRA η. In this simple economy, the price-dividend ratio is increasing in expected dividend growth if and only if $\eta < 1$. [10] Moreover, in the same economy, the unit risk-premia are such that $\lambda_2 = 0$ and $\lambda_1 = \eta \sigma_0$. With these parameters, eq. (14) *never* holds, and price-dividend ratios are linear in y - a feature consistent with Veronesi's results. However, had the author come up with a pricing kernel such that

$$\text{For all } y \in \mathbb{Y}, \quad R'\left(y\right) < \frac{\lambda_1}{\sigma_0} + \frac{\lambda_2}{\sigma_1}, \tag{15}$$

eq. (14) would hold, and price-dividend ratios would be convex in expected dividend growth.

The main lesson emanating from eqs. (14) and (15) is that in economies with switching-regimes, incomplete information and Bayesian learning mechanisms, convexity properties of pricing functions can be destroyed by excessive variability of interest rates and/or low levels of risk-premia. Intuitively, interest rates and dividend growth have opposite effects on the asset price. The same nonlinearities arising from the learning process may well affect interest rate behavior. A natural way to reduce interest rates volatility is to introduce a wedge between the asset's dividends and total consumption, and make consumption independent of dividend growth. However, in this case there are no linear signal structures and representative CRRA economies with *complete securities markets* leading to eq. (14), and thus supporting the convexity property. Appendix B ("Linear regime-switching economies") provides a proof of this claim. Finally, one may simply consider economies with elastically supplied safe assets - and hence pricing kernels with constant interest rates. In this case, a direct application of the conditions in eq. (14) reveals that price-dividend ratios are *always* convex under strictly positive CRRA.

The test conditions in eq. (14) formally describe how the effects of learning mechanisms impinge upon the equilibrium price process. It is thus very relevant to further discuss these conditions. Consider again the regime-switching models in example 1. Intuitively, these models predict that risk-aversion correction is almost nil during extreme situations (i.e. when expected dividend growth volatility is close to zero); and that it is at its highest during relatively more "normal" situations. This is so because rational agents would demand a very small risk-compensation for the fluctuation of expected dividend growth when expected dividend growth volatility is very low. More formally, the instantaneous (risk-adjusted) expected changes in dividend growth [that is, the risk-adjusted drift of y in eq. (9)] is $\hat{m}(y) = m(y) - \lambda_1 v_1(y) - \lambda_2 v_2(y)$, and is convex in y because v_1 and v_2 are inverse-U shaped. Eq. (14) now tells us that the price-dividend ratio inherits these same convexity properties of the instantaneous (risk-adjusted) expected changes in dividend growth. But as also emphasized earlier, these convexity effects can be offset by time-varying interest rates.

The previous effects on inherited convexity can be seen at work even more simply within model (10). Suppose as in Veronesi (1999) that the safe asset is elastically supplied so that the interest rate $R = r$, a constant. In a representative agent economy with CARA, the asset price is, by eq. (3),

$$P(D,y) = \int_0^\infty C(D,y,t)\, dt, \quad C(D,y,t) \equiv e^{-rt}(D - \sigma_0 \lambda_1 t) + e^{-rt} \int_0^t \mathbb{E}(y_u \mid y)\, du.$$

By the previous equation, $P(D,y)$ is increasing and convex in the current expected dividend growth y if the conditional expectation of future expected dividend growth $\mathbb{E}(y_u \mid y)$ is increasing

16

and convex in y. But according to lemma 1, the conditional expectation $\mathbb{E}\left(y_u \middle| y\right)$ is convex in y whenever the risk-adjusted drift $\hat{m}\left(y\right)$ is convex in y. By the previous reasoning on risk-adjustement issues during extreme situations, this risk-neutral drift $\hat{m}\left(y\right)$ is convex in y. [11] Therefore, convexity of the asset price is inherited by convexity of the risk-neutralized expected changes in dividend growth.

If fundamentals are not subject to switching-regimes (as in example 1), but information disseminated in the economy is still incomplete, Bayesian learning mechanisms may produce quite different effects on the asset price function. Consider example 2, in which fundamentals have unbounded support. This example is closely related to Brennan and Xia (2001) model. In their article, Brennan and Xia (2001) originally introduced a wedge between consumption and dividends, thereby making interest rates constant. In this case, eq. (14) would still be relevant to address convexity issues. And it would predict that price-dividend ratios are *always* convex in y - i.e. *independently of risk-aversion and all other parameters in the model*. If interest rates are time-varying, the situation is similar. For instance, in the economy of example 2 with complete markets and a representative agent with CRRA η, an application of proposition 1 leads to the following results: price-dividend ratios are *a) always convex* in expected dividend growth y; and *b)* increasing (resp. decreasing) in expected dividend growth y if $\eta < 1$ (resp. $\eta > 1$).

As emphasized in the introduction, the central message in this section is that convexity properties of the price-dividend ratio are inherited by convexity properties of the risk-neutral changes in expected dividend growth - but can be destroyed by excessive variability of interest rates. I now apply proposition 1 to a new model example to find conditions under which the price-dividend ratio can even be *concave* in expected dividend growth. Consider the following special case of eqs. (8): Consumption endowment D_t is solution to,

$$
\begin{cases}
\dfrac{dD_t}{D_t} = y_t dt + \sigma_0 dW_{1t}, \qquad \sigma_0 > 0 \\[4mm]
dy_t = \kappa \left(a - y_t\right) y_t dt + \bar{v}_1 y_t^{3/2} dW_{1t} + \bar{v}_2 y_t^{3/2} dW_{2t}
\end{cases}
\tag{16}
$$

for four positive constants $\kappa, a, \bar{v}_1$ and $\bar{v}_2$. [12] Assume, also, that there exists a representative agent with CRRA η, and that she has access to a system of complete markets. It is a simple matter to apply proposition 1 to this model example. Simple computations (in appendix B) lead to the following conclusion: the price-dividend ratio is increasing in expected dividend growth y if $\eta < 1$, and: *a)* convex in y if $\bar{v}_1 > 0$ and $1 - \eta > \kappa$; *b)* concave in y if $\bar{v}_1 < 0$ and $1 - \eta < \kappa$; *c)* convex (resp. concave) in y if $\bar{v}_1 > 0$ (resp. < 0) and $1 - \eta = \kappa$. As it stands, the empirical evidence on the sign of the correlation between consumption and consumption growth is ambiguous [see, e.g.,

the discussion in Campbell (2003, p. 842)]. Interestingly, proposition 1 reveals that within model (16), the sign of $\bar{v}_1$ (and hence of this correlation) plays a critical role in determining convexity properties of the price-dividend ratio.

3.2 Time-varying discount rates

This section develops a theory analyzing the joint behavior of time-varying discount rates, asset returns and volatility. To isolate the effects of time-varying discount rates on asset prices, I assume that total consumption endowment $\{D_t\}_{t \geq 0}$ is generated by a simple geometric Brownian motion

$$\frac{dD_t}{D_t} = g_0 dt + \sigma_0 dW_{1t}, \tag{17}$$

where g_0 and σ_0 are positive constants. The unit-risk premia λ_i are assumed to be functions of one additional state variable y only. Consequently I set,

$$\lambda_i(y) \equiv \lambda_i(D, y) \quad i = 1, 2,$$

where $\{y_t\}_{t \geq 0}$ is solution to,

$$dy_t = m(y_t)dt + v_1(y_t)dW_{1t} + v_2(y_t)dW_{2t}. \tag{18}$$

In this model, the state variable y_t drives variations in the risk-premia λ_i. In many cases of interest, y_t is a state variable related to the business cycle conditions, and $\partial \lambda_i / \partial y < 0$ (see example 3 below). In the same cases, the functional form of λ_i is easily deduced from first principles. On the other hand, the functional form of m, v_1 and v_2 is typically "variation free" - i.e. it is *not restricted* by standard asset pricing theories. In this section, I develop joint restrictions on m, v_i $(i = 1, 2)$ and λ_i $(i = 1, 2)$ that are consistent with properties of the pricing function $P(D, y)$ given in advance. For example, in the introduction and section 2, I argued that returns volatility is countercyclical if the two sources of fundamentals' volatility (v_1 and v_2) are constant, and P is concave in y. But how can we ensure that P is concave in y in this and more complex situations (with possible non constant v_i)? The conditions in this section explicitly address this issue. I now provide examples of models covered by the framework of this section.

Example 3. [Campbell and Cochrane (1999)]. Consider an infinite horizon, complete markets economy in which the representative agent has (undiscounted) instantaneous utility given by $u(c, x) = \left[(c - x)^{1-\eta} - 1 \right] / (1 - \eta)$, where c is consumption and x is a (time-varying) habit, or

18

(exogenous) "subsistence level". In equilibrium $c_t = D_t$, all t. Let $y \equiv (D-x)/D$ (the equilibrium "surplus consumption ratio"). By assumption, y_t is solution to:

$$dy_t = y_t \left[(1-\phi)(\bar{s} - \log y_t) + \frac{1}{2}\sigma_0^2 l(y_t)^2 \right] dt + \sigma_0 y_t l(y_t) dW_{1t}, \quad |\phi| < 1, \ \bar{s} \in \mathbb{R}, \ \sigma_0 > 0. \quad (19)$$

where l is a positive function given in appendix B. The Sharpe ratio predicted by the model is:

$$\lambda(y) = \eta\sigma_0 \left[1 + l(y) \right]$$

(see appendix B for additional details).

Time variation in the Sharpe ratio may also arise in economies where agents have time-separable preferences, but face an incomplete market structure. In these cases, Sharpe ratios are typically driven by state variables positively related to the utility of market participants. As an example, Basak and Cuoco (1998) model of restricted stock-market participation predicts that the Sharpe ratio $\lambda(y) \propto y^{-1}$, where now y is market participants' consumption share. Finally, Sharpe ratios may be time-varying simply because the public sector affects equilibrium conditions, as in the following example.

Example 4. (Government spending). Consider an infinite horizon, representative CRRA agent economy in which consumption endowment is solution to,
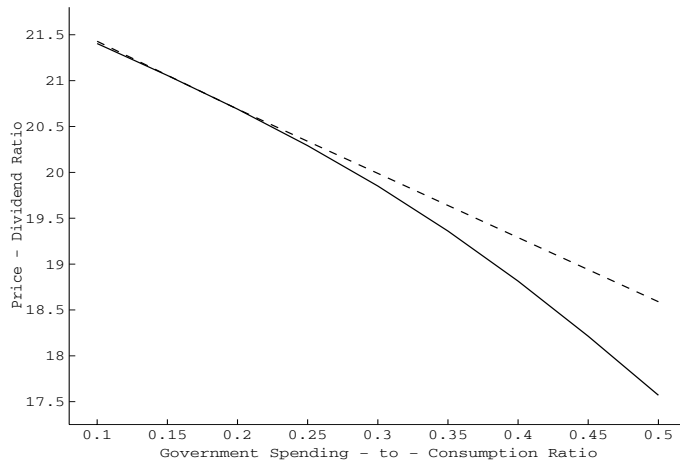
$$\frac{dD_t}{D_t} = g_0 dt + \sigma_0 dW_{1t}, \quad g_0, \sigma_0 > 0.$$

Let y_t be the ratio of Government spending to consumption endowment D_t. That is, let $Gvt_t \equiv y_t \cdot D_t$ $(t \geq 0)$, where Gvt is total Government spending. The ratio y_t is solution to:

$$dy_t = m\left(y_t\right) dt + v\left(y_t\right) dW_{2t},$$

where functions m and v are such that y_t is constrained to live in $\left(\underline{b}, \bar{b}\right)$ for some constants $\underline{b}, \bar{b} \in (0, 1)$. Finally, Government spending is financed by debt and non-distortionary taxes, and agents have access to a system of complete state-contingent markets. In equilibrium, aggregate consumption equals $\left(1 - y_t\right) D_t$ for all t.

The ratio of total Government spending over GDP (Gvt-ratio, henceforth) displays low volatility in many industrialized countries. Therefore, the Gvt-ratio can hardly explain high frequency

Figure 2 – Government size and asset prices

In the economy of example 4 and eq. (20), price-dividend ratios are decreasing and concave in y (the ratio of Government spending-to-consumption) (solid line). The dotted line is the linear approximation of price-dividend ratios at $y = 0.20$. Proposition 2 reveals that these properties arise because 1) the short-term rate is increasing and convex in y; and 2) the Government spending risk-premium is "sufficiently" convex in y.

movements in the stock-market. Yet this ratio is random, mean-reverting and highly persistent in the same countries. Despite its low volatility, the Gvt-ratio is a source of economic uncertainty for the long-run. The model in example 4 is new, and isolates the asset pricing implications of this source of uncertainty. It thus represents an interesting laboratory to study the accuracy of the theory in this section. [13]

Figure 2 depicts the numerical solution of the model obtained under the assumption that y_t is solution to,

$$dy_t = \beta\left(\bar{y} - y_t\right)dt + \sigma_1\left(\bar{b} - y_t\right)\left(y_t - \underline{b}\right)dW_{2t}, \quad \beta, \sigma_1 > 0, \tag{20}$$

where $\bar{y} \in \left(\underline{b}, \bar{b}\right)$, and $\underline{b}, \bar{b}$ are as in example 4. [14] As for all the models considered in this section, this model predicts that the price-dividend ratio is independent of D. Figure 2 also suggests that the price-dividend ratio is *decreasing and concave* in y. The economic interpretation of the monotonicity result is very simple: Mean reversion implies that high values of y_t (say) are expected to return to lower levels in the future, and this creates relatively less incentives to save. Consequently, as y_t increases interest rates increase and asset prices decrease. Understanding the

origins of the concavity property is also important and at the same time more challenging, even in this simple model. As an example, the magnitude of stochastic fluctuations in the price-dividend ratio doubles as the Gvt-ratio moves from the [10%, 15%] range to the [40%, 45%] range. Yet there is no obvious economic intuition that helps to explain this pattern.

The previous examples 3 and 4 very clearly illustrate that properties of models depend critically on the assumptions made on the primitives of the economy. For example, Campbell and Cochrane assumed that function l in (19) is positive, decreasing and convex over the relevant range of variation of y_t. Remarkably, their model makes the intriguing predictions that price-dividend ratio are concave in y, and that expected returns are both countercyclical. Yet what is the precise mechanism linking convexity of risk-adjusted discount rates, concavity of price-dividend ratios and countercyclical risk-premia and volatility? Or, why are price-dividend ratios *concave* in the Government spending-to-consumption ratio in Figure 2? The following proposition provides a theory addressing these questions in great generality.

Proposition 2. *Let the endowment process be as in eq. (17), and let interest rates and unit-risk premia be such that $R(D,y) = R(y)$ and $\lambda_i(D,y) = \lambda_i(y)$ ($i = 1,2$), where $\{y_t\}_{t\geq 0}$ is solution to eq. (18). Then, the price function $P(D,y) = D \cdot p(y)$, where p is a positive function satisfying the following properties*:

a) Suppose that $\forall y \in \mathbb{Y}$, $R'(y) + \sigma_0\lambda_1'(y) < 0$ (resp. > 0). Then, p is increasing (resp. decreasing).

b.1) Assume that p is increasing, and that $\forall y \in \mathbb{Y}$, $R''(y) + \sigma_0\lambda_1''(y) > 0$ (resp. ≤ 0) and $T(y) \equiv \frac{d^2}{dy^2}[m(y) - \sum_{i=1}^{2} \lambda_i(y)v_i(y) + \sigma_0 v_1(y)] - 2\left(R'(y) + \sigma_0\lambda_1'(y)\right) < 0$ (resp. > 0). Then, p is concave (resp. convex).

b.2) Assume that p is decreasing, and that $\forall y \in \mathbb{Y}$, $R''(y) + \sigma_0\lambda_1''(y) > 0$ (resp. ≤ 0) and $T(y) > 0$ (resp. < 0). Then, p is concave (resp. convex).

c) Suppose that $y \in \left(\underline{y}, \bar{y}\right)$, for some constants $\underline{y}$ and $\bar{y}$, and that $p' > 0$ for all $y \in \left(\underline{y}, \bar{y}\right)$, and $\lim_{y \to \underline{y}}\left(R(y) + \sigma_0\lambda_1(y)\right) = \infty$. Then, under technical regularity conditions in the appendix (conditions H1), there exists a $y^* > \underline{y}$ such that p is concave for all $y \in \left(\underline{y}, y^*\right)$.

Similarly as for proposition 1, parts a and b of proposition 2 can be understood with the help of lemma 1. Consider the evaluation formula in eq. (6). Under the assumption of this section, the price-dividend ratio satisfies,

$$p(y) = \int_0^\infty B(y,t)dt, \quad B(y,t) \equiv \bar{\mathbb{E}}\left[\exp\left(-\int_0^t \left(R(y_u) + \sigma_0\lambda(y_u) - g_0\right)du\right)\bigg|y\right], \qquad (21)$$

where $\bar{\mathbb{E}}$ is the expectation operator taken under a new measure $\bar{Q}$, and y_t is solution to

$$dy_t = \left[m(y_t) - \sum_{i=1}^{2} \lambda_i(y_t)v_i(y_t) + \sigma_0 v_1(y_t) \right] dt + v_1(y_t)\, d\bar{W}_{1t} + v_2(y_t)\, d\bar{W}_{2t},$$

where $\bar{W}_1$ is a $\bar{Q}$-Brownian motions and $\bar{W}_2 = \hat{W}_2$. [15] According to eq. (21), the price-dividend ratio p is a linear functional of bond prices $\{B(y,t)\}_{t \geq 0}$ in a fictitious economy where the short-term rate is given by $\rho(y) \equiv R(y) + \sigma_0 \lambda(y) - g_0$. General properties of p in (21) may be deduced through an application of lemma 1 to function B. Monotonicity properties are straightforward. By lemma 1-a), B is increasing in y whenever risk-adjusted discount rates $R + \sigma_0 \lambda$ are decreasing in y. Convexity properties of p follow by lemma 1-b). For example, if B is decreasing in y, then it is also concave in y whenever $\frac{d^2}{dy^2}[m(y) + \sigma_0 v_1(y) - \sum_{i=1}^{2} \lambda_i(y)v_i(y)] < 2\rho'(y)$ and $\rho''(y) > 0$, all $y \in \mathbb{Y}$. By the definition of ρ, and by rearranging terms, these two inequalities are exactly the ones stated in proposition 2-b.1). The inequalities of proposition 2-b.2) follow similarly. Finally, part c) of proposition 2 can not be understood through lemma 1, but deserves a separate treatment (see appendix B). I shall provide economic intuition underlying this part below.

Discussion

It is useful (but not compulsory) to think of y_t as a state variable related to business cycle conditions that are relevant to stock-market participants - just as in example 3. Proposition 2-a) then formalizes a simple idea about risk-adjusted discount rates $R + \sigma_0 \lambda$: If these discount rates are countercyclical, price-dividend ratios are *automatically* procyclical. Asset pricing theory may be ambiguous about the sign of the derivative $R'(y)$. But as proposition 2-a) indicates, models making short-term rates R "too" procyclical may also entail counterfactual consequences (namely, countercyclical price-dividend ratios).

Proposition 2-b) contains a *second-order* analysis of the models covered in this section. As emphasized earlier [see eqs. (5) in section 2], concavity of the price-dividend ratio p plays a critical role in explaining cyclical properties of both expected (excess, percentage) returns $\mathcal{E}$ and returns volatility $\mathcal{V}$. For example, if the two sources of fundamentals' volatility (v_1 and v_2) are constant, the unit risk-premia are countercyclical (i.e. the derivatives $\lambda_i'(y) < 0$) and p is concave, then $\mathcal{V}$ and $\mathcal{E}$ are both countercyclical. The intuition behind this effect is simple: returns volatility increases on the downside if the price-dividend ratio is concave in some variable related to the state of the economy. When is p concave then? According to proposition 2-b.1), p is concave whenever discount rates $R + \sigma_0 \lambda$ are convex with a curvature "sufficiently" high to make $T < 0$. The economic intuition underlying this condition has been developed in the introduction (see

Figure 1), and so in the interest of space I will not repeat it here. It is instead of interest to use the previous predictions to shed light on the nonlinear models in examples 3 and 4.

Consider first the Campbell and Cochrane model (1999) in example 3. This model is extraordinarily complex. The authors demonstrated numerically that in their model, the price-dividend ratio is substantially linear in the surplus consumption ratio y when y is high. At the same time, the authors found that the price-dividend ratio is concave in y for sufficiently low values of y. It is this concavity property which makes this model extremely complex to study analytically. Proposition 2 predicts the occurrence of this property. Furthermore, it predicts that this property arises because the Sharpe ratio is convex in the surplus consumption ratio y (in fact, *infinitely convex* as the surplus ratio y gets smaller and smaller). Intuitively, Campbell and Cochrane assumed that agents have a very slow adjustment habit process. As a result, their surplus ratio y spends a considerable amount of time in its lower (fat) tail. By the (infinite) convexity of the Sharpe ratio, the Sharpe ratio (and hence the risk-neutralized discount rate) is then highly and persistently volatile during bad times. Even if the parameters values used by the authors imply that $T > 0$ when y is low, infinite convexity of Sharpe ratios occurring at $y = 0$ and persistence of y imply that the price-dividend ratio is concave for small values of y. This implication is exactly proposition 2-c).

Do habit models always predict that price-dividend ratios are concave in the surplus consumption ratio? Consider for example Menzly, Santos and Veronesi (2004) (MSV) model of habit formation. The authors considered the same representative agent economy in example 3, but assumed that the *inverse* of the surplus consumption ratio $i_t = y_t^{-1}$ (say) is solution to,

$$di_t = k\left(\bar{\imath} - i_t\right)dt - \alpha\left(i_t - l\right)\sigma_0 dW_{1t}, \quad i_0 \in \left(l, \infty\right), \tag{22}$$

where l $(l > 1)$, $k, \bar{\imath}$ and α are positive constants. MSV used this framework to develop a multiple trees model in which the share of every tree's dividend is random. The authors also studied the aggregate consumption asset price dynamics. For this asset, the price-dividend ratio was linear in the surplus consumption ratio y. This result hinged upon the assumption of logarithmic preferences (i.e., $\eta = 1$ in example 3). What happens to the MSV pricing function when the risk-aversion parameter $\eta \neq 1$? Table 1 summarizes predictions emanating from proposition 2-a) and 2-b.1).

Once again, the economic intuition underlying the restrictions in Table 1 is related to convexity of risk-adjusted discount rates. In this model, the unit risk-premium λ_1 is decreasing and linear in the surplus consumption y. But the interest rate R is decreasing and *nonlinear* in y. Precisely, risk-adjusted discount rates $R(y) + \sigma_0 \lambda_1(y)$ are always decreasing in y; and convex (resp. concave)

Table 1

Price-dividend ratios and surplus consumption y. Predictions related to the MSV model.

parameter restriction	price-dividend ratio
$\eta \in (0,1)$ and $k\bar{\imath} > \alpha l\sigma_0^2 (\alpha + \eta - 1)$	increasing and *concave* in y
$\eta = 1$	increasing and *linear* in y
$\eta > 1$ and $k\bar{\imath} > \alpha l\sigma_0^2 \max(\alpha + \eta - 1, (1+\alpha)(\eta - 1))$	increasing and *convex* in y

in y if $\eta < 1$ (resp. $\eta > 1$) (see appendix B for a proof of this claim and Table 1 restrictions). Intuitively, the interest rate is decreasing in y because of intertemporal substitution effects. Moreover, precautionary effects kick in. And these precautionary effects are more and more severe as η increases: As η increases, the interest rate decreases by becoming concave in the surplus ratio y. Under the additional conditions in Table 1, this concavity translates to convexity of the price-dividend ratio.

General properties of the model in example 4 can be understood in a very similar manner. For this model, parts 2-a) and 2-b.2) are the relevant portions of proposition 2 to use to predict the occurrence of the concavity property in Figure 2. Precisely, the test conditions in proposition 2 reveal that in this model, the price-dividend ratio is decreasing and concave in the Gvt-ratio y whenever the interest rate R is increasing and convex in y, and the curvature of the Gvt-ratio risk premium is sufficiently high (that is, if $\frac{d^2}{dy^2}(-v(y)\lambda_2(y)) - 2\frac{d}{dy}R(y) < 0$, where λ_2 is the Gvt-ratio risk premium). In appendix B, I derive the equilibrium interest rate R and risk-premia for this model. For sake of completeness, I then apply the theoretical test conditions of proposition 2 and demonstrate that they do lead to the concavity result in Figure 2. In particular, both the interest rate R and the Gvt-ratio risk premium are convex in the Gvt-ratio y - and convexity of R originates from the same precautionary effects arising within the model in example 3.

4 Extensions

This section considers higher dimensional extensions of the theory. I take as primitive a general diffusion state process. I then restrict it to guarantee that all resulting asset price processes are consistent with given sets of properties. I consider the following environment. Let dividends

$\{D_t\}_{t \geq 0}$, the pricing kernel $\{\xi_t\}_{t \geq 0}$ and a three-dimensional state process $\{y_t\}_{t \geq 0}$ satisfy,

$$
\begin{cases}
dD_t = G\left(D_t, y_t\right) dt + \sigma\left(D_t\right) dW_{1t} \\
\\
d\xi_t = -\xi_t \left[R(D_t, y_t) dt + \sum_{j=1}^{4} \lambda_j(D_t, y_t) dW_{jt} \right] \\
\\
dy_{it} = m^{(i)}(D_t, y_t) dt + \sum_{j=1}^{4} v_j^{(i)}(y_t) dW_{jt} \qquad i = 1, 2, 3
\end{cases}
\tag{23}
$$

where $y = (y_1, \ y_2, \ y_3)^\top$ and $\{W_j\}_{j=1}^{4}$ are independent Brownian motions. Accordingly, the no-arbitrage price function is $P(D, y) \in \mathcal{C}^{2,2,2,2}(\mathbb{D} \times \mathbb{Y})$, $\mathbb{Y} \subset \mathbb{R}^3$. Furthermore, functions G, σ, R, λ_j, $m^{(i)}$, $v_j^{(i)}$ ($i = 1, 2, 3$ and $j = 1, \cdots, 4$) satisfy the same conditions as those of eqs. (2) and (4) in section 1. Finally, let $\bar{v}_i^2 \equiv \left\| (v_1^{(i)}, \cdots, v_4^{(i)}) \right\|^2$ and $v^{h,k} \equiv \sum_{j=1}^{4} v_j^{(h)} v_j^{(k)}$.

In this model, asset prices variations originate from the fluctuation of four factors: dividends (D_t); and three additional state variables affecting expected dividend growth (G), risk-premia (λ_j) and the short-term interest rate (R). This formulation allows expected dividend growth, risk-premia and the short-term interest rate to be imperfectly correlated - even when risk-premia and the short-term rate do not depend on D. Brennan, Wang and Xia (2003) have recently considered specific cases of system (23), and found closed-form solutions for the pricing function $P(D, y)$. Even the *individual* stock prices in the economies considered in Menzly, Santos and Veronesi (2004) can be thought of as being generated by a specific mechanism that is similar to system (23). Here I aim at providing a general asset pricing characterization relying on as few assumptions as possible as regards the primitive dynamics.

Similarly as in section 3, I now only consider scale-invariant economies in which the price-dividend ratio is independent of D. The price-dividend ratio is indeed independent of D if a) R, λ_j and $m^{(i)}$ are independent of D; and b) D_t is is solution to,

$$
\frac{dD_t}{D_t} = g(y_{1t}, y_{2t}, y_{3t}) dt + \sigma_0 dW_{1t}, \sigma_0 > 0,
\tag{24}
$$

where function g is twice differentiable in all its arguments (see proposition 3 below). [16] Finally, I set $\hat{g}(y) \equiv g(y) - \sigma_0 \lambda_1(y)$. Similarly, functions $\hat{m}^{(i)}(y) \equiv m^{(i)}(y) - \sum_{i=1}^{4} \lambda_j(y) v_j^{(i)}(y)$ denote risk-neutralized drift functions.

We have:

Proposition 3. *Assume that the instantaneous dividend rate is solution to eq. (24), that the pricing kernel is as in eqs. (23), and that the short-term rate R, unit risk-premia λ_j, and $m^{(i)}$ in (23) are all independent of D. Then, the price function $P(D,y) = D \cdot p(y)$, where the price-dividend ratio p is positive, and satisfies the following properties:*

a) Suppose that $\bar{v}_i^2$, $\hat{m}^{(i)}$, $v_1^{(i)}$ ($i = 2, 3$) and $v^{2,3}$ are independent of y_1. Then, p is increasing (resp. decreasing) in y_1 whenever $\frac{\partial}{\partial y_1}[\hat{g}(y) - R(y)] > 0$ (resp. < 0) for all y.

b) Suppose that $\bar{v}_i^2$, $\hat{m}^{(i)}$, $v_1^{(i)}$ ($i = 2, 3$), $v^{2,3}$ are independent of y_1 and that $\frac{\partial^2}{\partial y_1^2} v^{1,2}(y) = \frac{\partial^2}{\partial y_1^2} v^{1,3}(y) = 0$ for all y. Then, if p is increasing in y_1, it is also concave (resp. convex) in y_1 if $\frac{\partial^2}{\partial y_1^2}[\hat{g}(y) - R(y)] \leq 0$ (resp. ≥ 0) and $2\frac{\partial}{\partial y_1}\hat{g}(y) + \frac{\partial^2}{\partial y_1^2}\hat{m}^{(1)}(y) + \sigma_0 \frac{\partial^2}{\partial y_1^2} v_1^{(1)}(y) - 2\frac{\partial}{\partial y_1} R(y) < 0$ (resp. > 0) for all y.

c) Suppose that for all y, p is increasing in y_3, and increasing and (weakly) convex in y_1 (i.e. $\frac{\partial}{\partial y_3} p(y) > 0$, $\frac{\partial}{\partial y_1} p(y) > 0$ and $\frac{\partial^2}{\partial y_1^2} p(y) \geq 0$); that for $i \neq j$, the volatility function $\bar{v}_i^2$ of factor y_i does not depend on factor y_j, the risk-neutralized drift of factor y_2 (i.e. $\hat{m}^{(2)}$) is independent of factors y_1 and y_3; and finally, that $\frac{\partial}{\partial y_1}[\hat{g}(y) - R(y)] + \frac{\partial^2}{\partial y_1 \partial y_3}(\hat{m}^{(3)}(y) + \sigma_0 v_1^{(3)}) > 0$; $\frac{\partial}{\partial y_3}[\hat{g}(y) - R(y)] + \frac{\partial^2}{\partial y_1 \partial y_3}(\hat{m}^{(1)}(y) + \sigma_0 v_1^{(1)}) > 0$, and $\frac{\partial^2}{\partial y_1 \partial y_3}[\hat{g}(y) - R(y)] \geq 0$. If p is weakly convex (resp. concave) in y_3 and $\frac{\partial}{\partial y_1}\hat{m}^{(3)}(y) \geq 0$ (resp. ≤ 0), then, for all y, $\frac{\partial^2}{\partial y_1 \partial y_3} p(y) > 0$.

A simple example illustrating proposition 3 is a *three*-factor model in which aggregate consumption $\{c_t\}_{t \geq 0}$ (say) is distinct from (although correlated with) the stream of dividends $\{D_t\}_{t \geq 0}$; the expected dividend growth is affected by a state variable $\{y_{1t}\}_{t \geq 0}$; the unit risk-premia are such that $\lambda_2 = \lambda_3 = \lambda_4 = 0$, $\lambda \equiv \lambda_1$, where the risk-premium λ is driven by a procyclical "risk-preferences process" $\{y_{3t}\}_{t \geq 0}$, and is such that $y_3 \mapsto \lambda(y_3)$ is differentiable and decreasing; and $(D_t, y_{1t}, c_t, y_{3t})$ $(c_t \equiv y_{2t})$ satisfy:

$$\begin{cases} \dfrac{dD_t}{D_t} = g(y_{1t})dt + \sigma_0 dW_{1t} \\[2mm] dy_{1t} = m^{(1)}(y_{1t})dt + v_1^{(1)}(y_{1t})dW_{1t} + v_2^{(1)}(y_{1t})dW_{2t} \\[2mm] dc_t = m^{(2)}(c_t)dt + v_1^{(2)}(c_t)dW_{1t} + v_3^{(2)}(c_t)dW_{3t} \\[2mm] dy_{3t} = m^{(3)}(y_{3t})dt + v_1^{(3)}(y_{3t})dW_{1t} + v_3^{(3)}(y_{3t})dW_{3t} \end{cases} \qquad (25)$$

Next, suppose that the interest rate R is a function of y_3 only. (This assumption is consistent with the standard prediction of many models such as models with external habit formation.) By proposition 3-a,b), the price-dividend ratio is a function $p(y_1, y_3)$ *a) increasing* and *convex* in

26

the state variable y_1 affecting expected dividend growth whenever $\frac{d}{dy_1}g(y_1) > 0$, $\frac{d^2}{dy_1^2}g(y_1) \geq 0$ and $2\frac{d}{dy_1}g(y_1) + \frac{\partial^2}{\partial y_1^2}[m^{(1)}(y_1) + (\sigma_0 - \lambda(y_3))v_1^{(1)}(y_1)] > 0$ all (y_1, y_3); and $b)$ *increasing* in the risk-preferences state variable y_3 if $-\sigma_0\frac{d}{dy_3}\lambda(y_3) - \frac{d}{dy_3}R(y_3) > 0$, and *concave* in y_3 whenever $\frac{d^2}{dy_3^2}(\lambda(y_3) + \sigma_0 R(y_3)) > 0$ and $\frac{\partial^2}{\partial y_3^2}[m^{(3)}(y_1) + (\sigma_0 - \lambda(y_3))v_1^{(3)}(y_1)] - 2\sigma_0\frac{d}{dy_3}\lambda(y_3) - 2\frac{d}{dy_3}R(y_3) < 0$, all (y_1, y_3). These properties represent multidimensional extensions of those found and discussed in section 3, and are not discussed further here.

A new interesting aspect of multidimensional models is related to the price reaction to *joint* movements in the underlying state variables. Proposition 3-c) provides predictions on the direction of this reaction. Suppose for example that in model (25), risk-adjusted discount rates $R(y_3) + \sigma_0\frac{d}{dy_3}\lambda(y_3)$ are countercyclical, and that the dividend growth volatility are increasing in dividend growth (or that the slope of dividend growth volatility $\frac{d}{dy_1}v_1^{(1)}(y_1)$) is sufficiently low in absolute value. Under these mild conditions, proposition 3-c) predicts that it is *always* true that $p_{y_1 y_3} > 0$ in this model. In multidimensional models such as (25), changes in the expected dividend growth are very likely to have an even larger price impact than the one described in section 3: in addition to the convexity effects explained in section 3.1, price-dividend ratios and expected returns are now positively related also through these additional cross-effects. The economic significance underlying these magnifying effects is very important: an increase in expected dividend growth y_1 induces an increase in both the price-dividend ratio and expected price variations (and hence returns) in this model; therefore, price-dividend ratios can hardly be expected to be reliable predictors of future returns and dividend growth. Menzly, Santos and Veronesi (2004) were the first to make this point within an equilibrium model with habit formation and time-varying expected dividend growth. The results of this section reinforce their main conclusions: as proposition 3-c) reveals, a positive relation between expected dividend growth and expected returns is likely to emerge in large classes of economies under a set of rather mild conditions.

5 Conclusion

The basic *one-factor* Lucas (1978) asset pricing framework can considerably be enriched to allow for time-variation in both expected dividend growth and risk-adjusted discount rates. Such a research strategy has generated a new impetus in the literature. While the resulting models are making a real progress towards our understanding of aggregate stock-market behavior, the same models are often based on new assumptions concerning the dynamics of unobservable processes (such as time-varying expected dividend growth, or habit formation). As in many other asset

pricing contexts, the choice of these assumptions is typically guided by economically sensible intuition, casual empirical evidence, or analytical convenience. Yet each particular assumption should carry a critical weight on to the overall general properties of the resulting pricing functions. This article adds a new perspective and explores such general properties in a framework relying only on three basic assumptions: 1) asset prices are arbitrage free and homogeneous in the dividend process; 2) agents have rational expectations; and 3) state variables follow low-dimensional diffusion processes. Moreover, the methodology of investigation put forward in this paper can be extended to analyze (and design) models not strictly included in the class of economies I considered here. For example, it would be entirely feasible to extend the theory so as to cover the market-to-book ratio models introduced by Pástor and Veronesi in a series of papers [e.g., Pástor and Veronesi (2004)]; and multiple trees models.

The theoretical test conditions in this article enable one to understand properties of models directly from first principles. As a by-product, they explicitly investigate the robustness of these properties to the modification of "typical" assumptions; accordingly, I produced many examples illustrating how to apply the theory to shed new light on already existing models. Furthermore, these test conditions may be used as a practical guidance to specification, estimation and testing of new rational models. Consider for example a model making realistic predictions about the level of interest rates, equity premium and returns volatility. For concreteness, suppose that a sensible calibration of this model produces a pricing kernel quantitatively consistent with the Hansen and Jagannathan (1991) bounds. Would the same model be equally successful in explaining the typical business cycle variations in, say, the equity premium and returns volatility? Not necessarily. For example, this model may predict (counterfactually) that returns volatility is *pro*cyclical. The theory in this article provides conditions on models' primitives which generate the "right" volatility dynamics. Naturally, the approach pursued here does not stand as an alternative to usual model-testing protocols; for example, models that do satisfy the set of restrictions in this article require further quantitative scrutiny. Yet all the testing conditions developed here can be used to interpret the empirical success of failure of complex models that can only be solved through numerical methods or simplifying assumptions.

Finally, the theory in this article makes novel testable restrictions on the joint behavior of asset prices, risk-premia and the dynamics of consumption. For example, an important result in this paper is that countercyclical returns volatility can be activated by asymmetries in discounting occurring during the different phases of the business cycle. Whether an asymmetry in discounting exists in the first place is therefore a new crucial empirical issue for many rational explanations of asset price fluctuations.

Appendix A: Proofs for section 2

This appendix contains preliminary results. First, I assume the regularity conditions in section 1, and derive the Feynman-Kac stochastic representation of the partial derivatives of asset prices for the model in section 1: see lemma A1. Second, I use lemma A1 to find primitive conditions ensuring that assumption 1 in section 2 does hold. Third, I prove lemma 1. In all appendixes, subscripts denotes partial derivatives; for example, $P_{DD}(D, y) \equiv \frac{\partial^2}{\partial D^2} P(D, y)$, etc.

Lemma A1. *Let $w^1(D, y) \equiv P_D(D, y)$, $w^2(D, y) \equiv P_{DD}(D, y)$, $w^3(D, y) \equiv P_y(D, y)$, $w^4(D, y) \equiv P_{yy}(D, y)$ and $w^5(D, y) \equiv P_{Dy}(D, y)$. We have:*

$$w^i(D, y) = \mathbb{E}\left[\int_0^\infty \kappa_t^i h^i(\zeta_{it}, \gamma_{it}) dt\right], \quad i = 1, \cdots, 5,$$

where κ^i are random, positive processes defined in the proof,

$$h^1(D, y) = 1 + \hat{m}_D(D, y) P_y(D, y) - R_D(D, y) P(D, y)$$

$$h^2(D, y) = \left[\hat{G}_{DD}(D, y) - 2R_D(D, y)\right] P_D(D, y) - R_{DD}(D, y) P(D, y)$$
$$+ \hat{m}_{DD}(D, y) P_y(D, y) + \left[2\hat{m}_D(D, y) + \frac{\partial^2}{\partial D^2}((\sigma v_1)(D, y))\right] P_{Dy}(D, y)$$

$$h^3(D, y) = \hat{G}_y(D, y) P_D(D, y) - R_y(D, y) P(D, y)$$

$$h^4(D, y) = \hat{G}_{yy}(D, y) P_D(D, y) + \hat{m}_{yy}(D, y) P_y(D, y) + \left[2\hat{G}_y(D, y) + \frac{\partial^2}{\partial y^2}((\sigma v_1)(D, y))\right] P_{Dy}(D, y)$$
$$- R_{yy}(D, y) P(D, y) - 2R_y(D, y) P_y(D, y)$$

$$h^5(D, y) = \hat{G}_{Dy}(D, y) P_D(D, y) + \hat{G}_y(D, y) P_{DD}(D, y) + \hat{m}_{Dy}(D, y) P_y(D, y) + \hat{m}_D(D, y) P_{yy}(D, y)$$
$$- R_D(D, y) P_y(D, y) - R_{Dy}(D, y) P(D, y) - R_y(D, y) P_D(D, y)$$

and ζ_i, γ_i are solutions to some stochastic differential equations that are also given in the proof.

    Proof. In the absence of arbitrage, the price function $P(D, y)$ is solution to,

$$0 = \frac{1}{2}\sigma^2 P_{DD} + \hat{G} P_D + \frac{1}{2}\bar{v}^2 P_{yy} + \hat{m} P_y + \sigma v_1 P_{yD} + D - RP, \quad \forall (D, y) \in \mathbb{D} \times \mathbb{Y}, \tag{A1}$$

where $\bar{v}^2 \equiv v_1^2 + v_2^2$. By differentiating eq. (A1) with respect to D and y an appropriate number of times, I find that w^i are solutions to the following partial differential equations:

$$0 = (L^i - k^i)w^i(D,y) + h^i(D,y), \quad \forall(D,y) \in \mathbb{D} \times \mathbb{Y}, \quad i = 1,\cdots,5,$$

where $L^i w^i = \frac{1}{2}\sigma^2 w_{DD}^i + \hat{G}^i w_D^i + \frac{1}{2}\bar{v}^2 w_{yy}^i + \hat{m}^i w_y^i + \sigma v_1 w_{Dy}^i$, and

$$k^1(D,y) = R(D,y) - \hat{G}_D(D,y)$$

$$k^2(D,y) = R(D,y) - 2\hat{G}_D(D,y) - \frac{1}{2}\frac{\partial^2}{\partial D^2}(\sigma(D)^2)$$

$$k^3(D,y) = R(D,y) - \hat{m}_y(D,y)$$

$$k^4(D,y) = R(D,y) - 2\hat{m}_y(D,y) - \frac{1}{2}\frac{\partial^2}{\partial y^2}(\bar{v}^2(y))$$

$$k^5(D,y) = R(D,y) - \hat{G}_D(D,y) - \hat{m}_y(D,y) - \frac{\partial^2}{\partial D \partial y}(\sigma v_1)(D,y)$$

where I have defined, $\hat{G}^1 \equiv \hat{G} + \frac{1}{2}\frac{\partial \sigma^2}{\partial D}$, $\hat{m}^1 \equiv \hat{m} + \frac{\partial}{\partial D}(\sigma v_1)$, $\hat{G}^2 \equiv \hat{G} + \frac{\partial \sigma^2}{\partial D}$, $\hat{m}^2 \equiv \hat{m} + 2\frac{\partial}{\partial D}(\sigma v_1)$, $\hat{G}^3 \equiv \hat{G} + \frac{\partial}{\partial y}(\sigma v_1)$, $\hat{m}^3 \equiv \hat{m} + \frac{1}{2}\frac{\partial \bar{v}^2}{\partial y}$, $\hat{G}^4 \equiv \hat{G} + 2\frac{\partial}{\partial y}(\sigma v_1)$, $\hat{m}^4 \equiv \hat{m} + \frac{\partial \bar{v}^2}{\partial y}$, $\hat{G}^5 \equiv \hat{G} + \frac{1}{2}\frac{\partial \sigma^2}{\partial y} + \frac{\partial}{\partial y}(\sigma v_1)$, $\hat{m}^5 \equiv \hat{m} + \frac{1}{2}\frac{\partial \bar{v}^2}{\partial y} + \frac{\partial}{\partial D}(\sigma v_1)$. The result then follows by the Feynman-Kac probabilistic representation theorem: processes κ^i are given by $\kappa_t^i \equiv \exp(-\int_0^t k^i(\zeta_{iu}, \gamma_{iu})du)$, where ζ_i and γ_i are solutions to

$$\begin{cases} d\zeta_{it} = \hat{G}^i(\zeta_{it}, \gamma_{it})dt + \sigma(\zeta_{it})d\hat{W}_{1t} \\ \\ d\gamma_{it} = \hat{m}^i(\zeta_{it}, \gamma_{it})dt + v_1(\gamma_{it})d\hat{W}_{1t} + v_2(\gamma_{it})d\hat{W}_{2t} \end{cases}$$

with $(\zeta_{i0}, \gamma_{i0}) = (D,y)$, for $i = 1,\cdots,5$. ∎

Scale-invariant economies. Here I provide a proof of a claim made in section 2: Let G, σ, m in eqs. (2) satisfy $G(D,y) = g(y)D$, $\sigma(D) = \sigma_0 D$, for some constant $\sigma_0 > 0$ and some function g, and $m(y) \equiv m(D,y)$; and let $R(y) \equiv R(D,y)$ and $\lambda_i(y) \equiv \lambda_i(D,y)$ $(i = 1,2)$ in eqs. (2). Then, assumption 1 in section 2 holds true. To demonstrate this claim, I use the previous lemma A1 and conclude that $P_{DD} = 0$ whenever

$$h^2(D,y) = \left[\hat{G}_{DD}(D,y) - 2R_D(D,y)\right]P_D(D,y) - R_{DD}(D,y)P(D,y)$$
$$+ \hat{m}_{DD}(D,y)P_y(D,y) + \left[2\hat{m}_D(D,y) + \frac{\partial^2}{\partial D^2}((\sigma v_1)(D,y))\right]P_{Dy}(D,y) = 0,$$

which it does if G, σ, R, and λ are as prescribed. Hence P necessarily satisfies $P(D, y) \equiv p_o(y) + p(y)D$, for some functions p_o and p. By replacing this function into eq. (A1) leaves,

$$\forall (D, y) \in \mathbb{D} \times \mathbb{Y}, \quad 0 = \left\{ \frac{1}{2} \bar{v}^2(y) p''(y) + [\hat{m}(y) + \sigma_0 v_1(y)] p'(y) - [R(y) - \hat{g}(y)] p(y) + 1 \right\} D$$

$$+ \frac{1}{2} \bar{v}^2(y) p_o''(y) + \hat{m}(y) p_o'(y) - R(y) p_o(y)$$

(A2)

where $\hat{m}(y) \equiv m(y) - \lambda_1(y) v_1(y) - \lambda_2(y) v_2(y)$. In particular, this implies that,

$$0 = \frac{1}{2} \bar{v}^2(y) p_o''(y) + \hat{m}(y) p_o'(y) - R(y) p_o(y), \quad \forall y \in \mathbb{Y}.$$

One solution for p_o is $p_o \equiv 0$. In fact, $p_o \equiv 0$ is the only solution compatible with no-bubbles.

Proof of lemma 1. Let $c(y, T - s) \equiv \mathbb{E}[\exp(-\int_s^T \rho(y_t)dt) \cdot \psi(y_T) \,|\, y_s = y]$. Function c is solution to the following partial differential equation:

$$\begin{cases} 0 = -c_2(y, T - s) + L^* c(y, T - s) - \rho(y)c(y, T - s), & \forall (y, s) \in \mathbb{R} \times [0, T) \\ \\ c(y, 0) = \psi(y), & \forall y \in \mathbb{R} \end{cases}$$

(A3)

where $L^* c(y, u) = \frac{1}{2} a(y)^2 c_{yy}(y, u) + b(y) c_y(y, u)$. By differentiating twice eq. (A3) with respect to y, I find that $c^{(1)}(y, t) \equiv c_y(y, t)$ and $c^{(2)}(y, t) \equiv c_{yy}(y, t)$ are solutions to the following partial differential equations: $\forall (y, s) \in \mathbb{R}_{++} \times [0, T)$,

$$\begin{aligned} 0 = & -c_2^{(1)}(y, T - s) + \frac{1}{2} a(y)^2 c_{yy}^{(1)}(y, T - s) + \left[b(y) + \frac{1}{2}(a(y)^2)' \right] c_y^{(1)}(y, T - s) \\ & - \left[\rho(y) - b'(y) \right] c^{(1)}(y, T - s) - \rho'(y)c(y, T - s), \end{aligned}$$

with $c^{(1)}(y, 0) = \psi'(y) \; \forall y \in \mathbb{R}$; and $\forall (y, s) \in \mathbb{R} \times [0, T)$,

$$\begin{aligned} 0 = & -c_2^{(2)}(y, T - s) + \frac{1}{2} a(y)^2 c_{yy}^{(2)}(y, T - s) + \left[b(y) + (a(y)^2)' \right] c_y^{(2)}(y, T - s) \\ & - \left[\rho(y) - 2b'(y) - \frac{1}{2}(a(y)^2)'' \right] c^{(2)}(y, T - s) \\ & - \left[2\rho'(y) - b''(y) \right] c^{(1)}(y, T - s) - \rho''(y)c(y, T - s), \end{aligned}$$

with $c^{(2)}(y, 0) = \psi''(y) \; \forall y \in \mathbb{R}$ (in both equations, subscripts denote partial derivatives). By arguments similar to the ones used to prove lemma A1, we have that $c^{(1)}(y, T - s) > 0$ (resp. < 0) $\forall (y, s) \in \mathbb{R} \times [0, T)$ whenever $\psi'(y) > 0$ (resp. < 0) and $\rho'(y) < 0$ (resp. > 0) $\forall y \in \mathbb{R}$. This completes the proof of part a) of the proposition. The proof of part b) is obtained similarly. ∎

Appendix B: Proofs for section 3

Proof of proposition 1. If $P(D, y) = D \cdot p(y)$, and the conditions in section 3.1, functions h^3 and h^4 in lemma A1 collapse to

$$h^3(D, y) = \left[\hat{g}'(y) - R'(y)\right] p(y) \cdot D \equiv \bar{h}^3(y) \cdot p(y)$$

$$h^4(D, y) = \left\{ \left[\hat{g}''(y) - R''(y)\right] p(y) + \left[\hat{m}''(y) + 2\left(\hat{g}'(y) - R'(y)\right) + \sigma_0 v_1''(y)\right] p'(y) \right\} \cdot D \equiv \bar{h}^4(y) \cdot D,$$

where for all y, $\hat{g}'(y) \equiv \frac{d}{dy}\left(g(y) - \sigma_0 \lambda_1\right) = g'(y)$. The stochastic representations for $p'(y)$ and $p''(y)$ are then as follows:

$$p'(y) = \mathbb{E}\left[\int_0^\infty \kappa_t^3 \bar{h}^3\left(\gamma_{3t}\right) dt\right]$$

and

$$p''(y) = \mathbb{E}\left[\int_0^\infty \kappa_t^4 \bar{h}^4\left(\gamma_{4t}\right) dt\right],$$

where κ^i and γ^i ($i = 3, 4$) are as in lemma A1. ∎

Proof of proposition 2 [Parts a and b]. I proceed similarly as in the proof of proposition 1. By eq. (A2) in appendix A, the price-dividend ratio $p(y)$ satisfies:

$$0 = \frac{1}{2}\bar{v}^2 p'' + (m + \sigma_0 v_1 - \lambda \cdot v) p' - (R - g_0 + \sigma_0 \lambda_1) p + 1. \tag{B1}$$

Hence, $p > 0$. Moreover,

$$p'(y) = \mathbb{E}\left[\int_0^\infty \kappa_t^3 \ell_1\left(\gamma_{3t}\right) dt\right],$$

and

$$p''(y) = \mathbb{E}\left[\int_0^\infty \kappa_t^4 \ell_2\left(\gamma_{4t}\right) dt\right],$$

where κ^i and γ^i ($i = 3, 4$) are as in lemma A1, and

$$\ell_1(y) \equiv -\left[R'(y) + \sigma_0 \lambda_1'(y)\right] p(y),$$

$$\ell_2(y) \equiv -\left[R''(y) + \sigma_0 \lambda_1''(y)\right] p(y) + \left[\frac{d^2}{dy^2}\left(m(y) + \sigma_0 v_1(y) - \lambda(y) \cdot v(y)\right) - 2\left(R'(y) + \sigma_0 \lambda_1'(y)\right)\right] p'(y)$$

with $\lambda(y) \cdot v(y) \equiv \lambda_1(y) v_1(y) + \lambda_2(y) v_2(y)$. ∎

Let $\hat{R}_t \equiv \mathcal{R}(y_t) \equiv R(y_t) + \sigma_0\lambda_1(y_t)$ $(t \geq 0)$ and $\bar{m}(y) \equiv m(y) + \sigma_0 v(y) - \sum_{i=1}^{2}\lambda_i(y)v_i(y)$. One set of technical regularity conditions required in the main text is:

H1: *Functions $\mathcal{R}$, $\bar{m}$ and v satisfy the following conditions*:

i: $\min_{y \in (\underline{y}, \bar{y})} (\mathcal{R}(y)) > g_0 > 0$

ii: $\lim_{y \to \underline{y}} \bar{m}(y) \geq 0$.

iii: $\lim_{y \to \underline{y}} \left[|\bar{m}(y)| + v_i(y)\right] \dfrac{|\mathcal{R}'(y)|}{\mathcal{R}(y)} < \infty$ $(i = 1, 2)$ *and* $\lim_{y \to \underline{y}} \bar{v}^2(y) \dfrac{|\mathcal{R}''(y)|}{\mathcal{R}(y)} < \infty$.

Condition H1-i is an integrability condition. Condition H1-ii requires that under measure $\bar{Q}$ introduced in the main text, y is mean reverting in a neighborhood of $\underline{y}$. Finally, condition H1-iii bounds the rate of explosion of $\mathcal{R}(y)$ to infinity at $\underline{y}$. I now prove part c of proposition 2.

Proof of proposition 2 [Part c]. By assumption, $\hat{R}$. is strictly positive. Hence $\hat{R}_t = \mathcal{R}(y_0) w_t(\alpha, \beta)$, $w_t(\alpha, \beta) \equiv \exp\left\{\int_0^t \left[\alpha(y_s) - \frac{1}{2}\left(\beta_1(y_s)^2 + \beta_2(y_s)^2\right)\right] ds + \int_0^t \sum_{i=1,2} \beta_i(y_s) d\bar{W}_{is}\right\}$, where $\bar{W}_i$ $(i = 1, 2)$ are $\bar{Q}$-Brownian motions and,

$$\alpha(y) \equiv \bar{m}(y) \frac{\mathcal{R}'(y)}{\mathcal{R}(y)} + \frac{1}{2}\bar{v}^2(y)\frac{\mathcal{R}''(y)}{\mathcal{R}(y)} \quad \text{and} \quad \beta_i(y) \equiv v_i(y)\frac{\mathcal{R}'(y)}{\mathcal{R}(y)}, \quad i = 1, 2.$$

The Feynman-Kac representation of p in eq. (B1) is,

$$p(y) = \bar{\mathbb{E}}\left[\int_0^\infty e^{g_0 - \int_0^t \mathcal{R}(y_u)du} dt\right] = \bar{\mathbb{E}}\left[\int_0^\infty e^{g_0 - \mathcal{R}(y)\int_0^t w_u(\alpha,\beta)du} dt\right].$$

where $\bar{\mathbb{E}}$ denotes expectation under $\bar{Q}$. By assumption, $\lim_{y \to \underline{y}} \mathcal{R}(y) = \infty$; by H1-i, $\min_{y \in (\underline{y}, \bar{y})} \mathcal{R}(y) > g_0$. Hence by dominated convergence and H1-iii, $\lim_{y \to \underline{y}} p(y) = 0$. By eq. (B1),

$$\frac{1}{2}\bar{v}^2 p'' = -1 + (\mathcal{R} - g_0)p - (m + \sigma_0 v_1 - \lambda \cdot v) p'. \tag{B2}$$

Now suppose that $\lim_{y \to \underline{y}} p'(y) = \infty$. This implies that $\lim_{y \to \underline{y}} p''(y) < 0$. Alternatively, assume that $\lim_{y \to \underline{y}} p'(y) < \infty$. By H1-ii, $\lim_{y \to \underline{y}}(m(y) + \sigma_0 v(y) - (\lambda \cdot v)(y)) \geq 0$. As demonstrated above, $\lim_{y \to \underline{y}} p(y) = 0$. Hence for small y, eq. (B2) is, $\frac{1}{2}\bar{v}^2 p'' \leq -1 + \mathcal{R}p$. In this case $\lim_{y \to \underline{y}} p''(y) < 0$ whenever $\lim_{y \to \underline{y}} \mathcal{R}(y) p(y) = 0$. But again $\inf_t (\mathcal{R}(y_t)) > g_0$, and the result follows by H1-iii and dominated convergence,

$$\lim_{y \to \underline{y}} \mathcal{R}(y) p(y)$$

$$= \lim_{y \to \underline{y}} \bar{\mathbb{E}}\left[\int_0^\infty e^{g_0 t}\mathcal{R}(y) e^{-\int_0^t \mathcal{R}(y_u)du} dt\right] = \lim_{y \to \underline{y}} \bar{\mathbb{E}}\left[\int_0^\infty e^{g_0 t}\left(\mathcal{R}(y) e^{-\mathcal{R}(y)\int_0^t w_u(\alpha,\beta)du}\right) dt\right] = 0. \quad \blacksquare$$

Linear regime-switching economies. Here I prove a claim made in section 3.1: Consider a complete markets economy in which dividends, consumption, and signals (D_t, c_t, a_t) satisfy:

$$
\begin{pmatrix} dD_t / D_t \\ dc_t / c_t \\ da_t \end{pmatrix} = \begin{pmatrix} \theta \\ \bar{g} \\ \theta \end{pmatrix} dt + \begin{pmatrix} \bar{\sigma}_0 \\ \bar{\sigma} \\ \bar{\sigma}_a \end{pmatrix} dw_t; \qquad
\begin{aligned}
\bar{\sigma}_0 &\equiv (\sigma_0 \ \ 0 \ \ 0) \\
\bar{\sigma} &\equiv (\sigma_1 \ \ \sigma_2 \ \ \sigma_3) \\
\bar{\sigma}_a &\equiv (\sigma_4 \ \ \sigma_5 \ \ \sigma_6)
\end{aligned}
$$

where $w = (w_1 \ w_2 \ w_3)^\top$ is a vector standard Brownian motion, θ is as in example 1, and $\bar{g}, \sigma_i$ are constants. Let $\sigma_3 \sigma_5 \neq \sigma_2 \sigma_6$. Then there are *no* CRRA representative agent equilibria in which price-dividend ratios are convex in expected dividend growth.

To demonstrate this claim, I first apply standard filtering results [Liptser and Shiryaev (2001) (Vol. I)], and find that the previous economy is isomorphic to one in which,

$$
\begin{pmatrix} dD_t / D_t \\ dc_t / c_t \\ dy_t \end{pmatrix} = \begin{pmatrix} y_t \\ \bar{g} \\ k(y^* - y_t) \end{pmatrix} dt + \begin{pmatrix} \bar{\sigma}_0 \\ \bar{\sigma} \\ (\bar{\theta} - y_t)(y_t - \underline{\theta}) \cdot \bar{\sigma}_y \end{pmatrix} dW_t,
$$

where $W = (W_1 \ W_2 \ W_3)^\top$ is a vector standard Brownian motion; $k, y^* > 0$; and $\bar{\sigma}_y$ is some vector satisfying $\bar{\sigma} \cdot \bar{\sigma}_y = 0$. Standard arguments lead that in equilibrium, R is constant and $\lambda_i \propto \sigma_i$ ($i = 1, 2, 3$). The claim follows by $\bar{\sigma} \perp \bar{\sigma}_y$, and an application of proposition 3 (Parts a and b).

Derivation of restrictions for model (16). By a standard argument, $R(y) = a + \eta y$ for some constant a. By proposition 1-a), $p' > 0$ whenever $\eta < 1$. By proposition 1-b), $p'' > 0$ (resp. < 0) whenever $\frac{3}{4} \bar{v}_1 \sigma_0 (1 - \eta) y^{-1/2} + 2(1 - \eta - \kappa) > 0$ (resp. < 0). Therefore, if $\eta < 1$, $p' > 0$ and: 1) for all $y \in \mathbb{Y}$, $p'' > 0$ if $\bar{v}_1 > 0$ and $1 - \eta > \kappa$; 2) for all $y \in \mathbb{Y}$, $p'' < 0$ if $\bar{v}_1 < 0$ and $1 - \eta < \kappa$; 3) for all $y \in \mathbb{Y}$, $p'' > 0$ (resp. < 0) if $\bar{v}_1 > 0$ (resp. < 0) and $1 - \eta = \kappa$.

Continuous time details for example 3. Campbell and Cochrane (1999) originally considered a discrete-time model. The diffusion limit of their consumption process is simply eq. (17) given in the main text. By standard martingale methods, one has

$$
\lambda(D, x) = \frac{\eta}{y} \left[\sigma_0 - \frac{1}{D} \gamma(D, x) \right], \tag{B3}
$$

where γ is the instantaneous volatility of $x = D(1 - y)$, and y is solution to eq. (19). By Itô's lemma, $\gamma = [1 - y - yl(y)] D\sigma_0$. By eq. (B3), $\lambda(y) = \eta \sigma_0 [1 + l(y)]$, as claimed in the main text. (This result only *approximately* holds in the original discrete time framework.) Finally, by another

application of martingale methods, $R(y) = \delta + \eta \left(g_0 - \frac{1}{2}\sigma_0^2 \right) + \eta(1-\phi)(\bar{s} - \log y) - \frac{1}{2}\eta^2\sigma_0^2 \left[1 + l(y) \right]^2$. R is constant whenever

$$l(y) = \frac{1}{\bar{S}}\sqrt{1 + 2(\bar{s} - \log y)} - 1, \quad y \in \left(0, \bar{S} \cdot e^{\frac{1}{2}(1-\bar{S}^2)} \right),$$

where $\bar{S} = \sigma_0\sqrt{\eta/(1-\phi)} = \exp(\bar{s})$. This corresponds to the same original restriction as in Campbell and Cochrane.

Derivation of Table 1 restrictions. If i is solution to eq. (22), the surplus ratio $y = i^{-1}$ is solution to eq. (18), with $m(y) = y[k(1 - \bar{\iota}y) + \alpha^2\sigma_0^2(1 - ly)^2]$, $v_1(y) = \alpha\sigma_0 y(1 - ly)$, and $v_2 = 0$. By standard martingale methods, R and λ can be computed to be,

$$
\begin{aligned}
R(y) &= \rho + \eta g_0 - \frac{1}{2}\sigma_0^2\eta(\eta + 1) + \eta k(1 - \bar{\iota}y) - \eta^2\sigma_0^2\alpha(1 - ly) - \frac{1}{2}\eta(1 - ly)^2\alpha^2\sigma_0^2(\eta - 1) \\
\lambda(y) &= \eta\sigma_0\left[1 + (1 - ly)\alpha\right]
\end{aligned}
$$

where ρ is the impatience rate.

Let $\mathcal{R}(y) \equiv R(y) + \sigma_0\lambda(y)$ and $\bar{m}''(y) \equiv m''(y) + \frac{d^2}{dy^2}\left\{[\sigma_0 - \lambda(y)]v(y)\right\}$. By differentiating,

$$\mathcal{R}'(y) = -\left(\theta - l\alpha^2\sigma_0^2\right)\eta + l^2\alpha^2\sigma_0^2\eta(1 - \eta)y, \quad \mathcal{R}''(y) = \eta\sigma_0^2\alpha^2 l^2(1 - \eta),$$

and

$$T(y) \equiv \bar{m}''(y) - 2\mathcal{R}'(y) = 2(\eta - 1)\left[\theta + (\eta - 3)l^2\alpha^2\sigma_0^2 y\right],$$

where $y \in \left(0, l^{-1}\right)$ and $\theta \equiv k\bar{\iota} + (1 - \eta)l\alpha\sigma_0^2 + (2 - \eta)l\alpha^2\sigma_0^2$. We now consider two cases arising when $\eta \in (0, 1)$ and $\eta > 1$ respectively.

Case a) ($\eta \in (0, 1)$). By proposition 2-a), $p' > 0$ whenever $\mathcal{R}' < 0$ for all $y \in \left(0, l^{-1}\right)$. In this case, it always holds that $\mathcal{R}' < 0$. The "concavity" claim in Table 1 follows by proposition 2-b.1). Indeed, in this case: *1)* it always holds that $\mathcal{R}'' > 0$; and *2)* $T < 0$ whenever $\theta + (\eta - 3)l^2\alpha^2\sigma_0^2 y > 0$ for all $y \in \left(0, l^{-1}\right)$ or whenever $\theta + (\eta - 3)l\alpha^2\sigma_0^2 > 0$; by rearranging terms, I find that $T < 0$ whenever $k\bar{\iota} > \alpha(\alpha + \eta - 1)l\sigma_0^2$.

Case b) ($\eta > 1$). In this case, $\mathcal{R}' < 0$ for all $y \in \left(0, l^{-1}\right)$ whenever $\nu \equiv k\bar{\iota} + (1 + \alpha)(1 - \eta)l\alpha\sigma_0^2 > 0$. So let $\nu > 0$. To prove the "convexity" claim in Table 1, note that in this case: *1)* it always holds that $\mathcal{R}'' < 0$; and *2)* $T > 0$ whenever $t(y) \equiv \theta + (\eta - 3)l^2\alpha^2\sigma_0^2 y > 0$ for all $y \in \left(0, l^{-1}\right)$. There are two cases to consider: *b1)* $\eta \in (1, 3)$: in this case $t > 0$ whenever $k\bar{\iota} > \alpha(\eta + \alpha - 1)l\sigma_0^2$; and *b2)* $\eta \geq 3$: In this case $t > 0$ whenever $\theta = \nu + l\alpha^2\sigma_0^2 > 0$, i.e. always. The claim follows by proposition 2-b.1).

Interest rates and risk-premia in example 4. By standard optimality conditions,

$$e^{-\delta t}\left[D_t\left(1-y_t\right)\right]^{-\eta} = l \cdot \exp\left[-\int_0^t \left(R\left(D_u, y_u\right) du + \frac{1}{2}\left\|\bar{\lambda}\left(D_u, y_u\right)\right\|^2\right) du - \int_0^t \bar{\lambda}\left(D_u, y_u\right) dW_u\right],$$

where l is a Lagrange multiplier, $W = (W_1\ W_2)^\top$, and $\bar{\lambda} = (\lambda_1\ \lambda_2)^\top$ is the risk-premia vector. By an application of Itô's lemma to both sides of the previous equation, and by identifying drift and diffusion terms, I find that $\lambda_1 \equiv \lambda_1(D, y) = \eta\sigma_0$, and

$$
\begin{aligned}
R(y) &\equiv R(D, y) = \delta + \eta g_0 - \frac{1}{2}\eta(\eta+1)\sigma_0^2 - \eta\frac{m(y)}{1-y} - \frac{1}{2}\eta(\eta+1)\left[\frac{v(y)}{1-y}\right]^2 \\
\lambda(y) &\equiv \lambda(D, y) = -\eta\frac{v(y)}{1-y}
\end{aligned}
$$

where $\lambda \equiv \lambda_2$. By proposition 2 (Part a), the price-dividend ratio $p : p' < 0$ whenever $R' > 0$. And by proposition 2 (Part b), $p'' < 0$ whenever for all $y \in (\underline{b}, \bar{b})$, $R'' > 0$ and

$$\ell_2(y) = -R''(y)p(y) + \kappa(y)p'(y) < 0; \quad \kappa(y) \equiv \frac{d^2}{dy^2}\left[m(y) - \lambda(y)v(y)\right] - 2R'(y).$$

Hence $p' < 0$ and $p'' < 0$ whenever $R' > 0$, $R'' > 0$, and $\kappa > 0$. If y is solution to eq. (20),

$$
\begin{aligned}
R(y) &= \delta + \eta g_0 - \frac{1}{2}\eta(\eta+1)\sigma_0^2 - \eta\beta\frac{\bar{y}-y}{1-y} - \frac{1}{2}\eta(\eta+1)\sigma_1^2\left[\frac{(\bar{b}-y)(y-\underline{b})}{1-y}\right]^2 \\
\lambda(y) &= -\eta\sigma_1\frac{(\bar{b}-y)(y-\underline{b})}{1-y}
\end{aligned}
$$

Given the parameter values used to produce figure 2 (see footnote 13), $R' > 0$ and $R'' > 0$; in this numerical application, $\kappa < 0$, which does not ensure that $\ell_2(y) < 0$ for all y. The test condition in proposition 2 (Part b) must be further elaborated. I require the following condition:

H2: *The risk-neutralized drift function of y is negatively sloped.*

The previous condition is satisfied in the numerical application in the main text, and arises in many related problems. I now show that under condition H2, $\ell_2 < 0$. By $p' < 0$,

$$\text{For all } y \in (\underline{b}, \bar{b}), \quad \ell_2(y) \leq -\left[R''(y) + C\kappa(y)\right]p(y), \quad C \equiv \max_{y \in (\underline{b}, \bar{b})}\left(-\frac{p'}{p}\right).$$

Therefore, it is sufficient to show that $R'' + C\kappa > 0$. I need to estimate C. Let $\hat{g}_0 \equiv g_0 - \eta\sigma_0^2$ and $q \equiv \max_{y \in (\underline{b}, \bar{b})} \{R''(y) [\beta(\bar{y} - y) - \eta v(y)\lambda(y)] + \frac{1}{2}R'''(y)v(y)^2\} + \max_{y \in (\underline{b}, \bar{b})} \{R''(y)v(y)v'(y)\}$. Clearly,

$$p(y) = \mathbb{E}\left[\int_0^\infty e^{\hat{g}_0 t - \int_0^t R(y_u)du}dt\right] \geq \int_0^\infty e^{-(R(\bar{b}) - \hat{g}_0)t}dt = \frac{1}{R(\bar{b}) - \hat{g}_0}.$$

Next, let $\omega_t \equiv \partial y_t / \partial y$ denote the sensitivity of the solution flow $\{y_t\}_{t \geq 0}$ to the initial condition $y_0 = y$. By Kunita (1990, thms. 4.71 and 4.72, p. 177), ω_t is solution to,

$$d\omega_t = \omega_t \cdot \left\{ \left[m'(y_t) - [v(y_t)\lambda(y_t)]'\right]dt + v'(y_t)d\hat{W}_{2t}\right\}, \quad \omega_0 = 1. \tag{B4}$$

Since $y \mapsto v(y)$ is bounded on $(\underline{b}, \bar{b})$ and, by condition H2, $t \mapsto \mathbb{E}(\omega_t)$ is bounded, we have,

$$-p'(y) = \mathbb{E}\left[\int_0^\infty e^{\hat{g}_0 t - \int_0^t R(y_u)du}\left(\int_0^t R'(y_u)\omega_u du\right)dt\right] \leq \int_0^\infty e^{-(R(\underline{b}) - \hat{g}_0)t}\left[\int_0^t \mathbb{E}\left(R'(y_u)\omega_u\right)du\right]dt,$$

where the inequality holds by $R' \geq 0$ and positivity of ω_t. Moreover, by eq. (B4), and $m' - (v\lambda)' < 0$ (condition H2),

$$\mathbb{E}(\omega_t) = 1 + \int_0^t \mathbb{E}\left\{\omega_u \left[m'(y_u) - [v(y_u)\lambda(y_u)]'\right]\right\}du \leq 1.$$

By Itô's lemma, condition H2, positivity of ω_t, and the previous inequality,

$$\mathbb{E}\left[\omega_t R'(y_t)\right] \leq R'(y) + q \cdot t.$$

Hence $-p'(y) \leq R'(y)(R(\underline{b}) - \hat{g}_0)^{-2} + q(R(\underline{b}) - \hat{g}_0)^{-3}$, and an estimate of C is function $\hat{C}$ defined as,

$$\hat{C}(y) \equiv \frac{\{R'(y)[R(\underline{b}) - \hat{g}_0] + q\}[R(\bar{b}) - \hat{g}_0]}{[R(\underline{b}) - \hat{g}_0]^3} \geq C.$$

Given the parameter values in the main text, $R''(y) + \hat{C}(y)\kappa(y) > 0$ for all $y \in (\underline{b}, \bar{b})$. Hence, $p'(y) < 0$ and $p''(y) < 0$ for all $y \in (\underline{b}, \bar{b})$, as I claimed in the main text.

Appendix C: Proofs for section 4

Proof of proposition 3 [Parts a and b]. By absence of arbitrage opportunities, $P(D, y_1, y_2, y_3)$ is solution to:

$$0 = (L_P - R(D, y_1, y_2, y_3))P(D, y_1, y_2, y_3) + D, \tag{C1}$$

37

where

$$L_P P = \frac{1}{2}\sigma^2 P_{DD} + \hat{G}P_D + \frac{1}{2}\sum_{i=1}^{3}\bar{v}_i^2 P_{y_i y_i} + \sum_{i=1}^{3}\hat{m}^{(i)}P_{y_i} + \sigma\sum_{i=1}^{3}v_1^{(i)}P_{Dy_i} + v^{1,2}P_{y_1 y_2} + v^{1,3}P_{y_1 y_3} + v^{2,3}P_{y_2 y_3}$$

and $\hat{G} \equiv G - \lambda\sigma$. (As also mentioned in the main text, the functions $\hat{m}^{(i)}$ denote risk-neutralized drifts.) Next, let $u^1 \equiv P_{y_1}$ and $u^2 \equiv P_{y_1 y_1}$. Functions u^i are solutions to,

$$0 = (L_j - k_j)\,u^j + h_j, \quad j = 1, 2, \tag{C2}$$

where $k_1 = R - \frac{\partial \hat{m}^{(1)}}{\partial y_1}$, $k_2 = R - 2\frac{\partial \hat{m}^{(1)}}{\partial y_1} - \frac{1}{2}\frac{\partial^2}{\partial y_1^2}\bar{v}_1^2$,

$$L_1\left[u^1\right] = L^3\left[u^1\right] + L_{11}\left[u^1\right]$$

$$L_2\left[u^2\right] = L^4\left[u^2\right] + L_{11}\left[u^2\right] + \frac{\partial v^{1,2}}{\partial y_1}b_{y_2} + \frac{\partial v^{1,3}}{\partial y_1}b_{y_3}$$

$$L_{11}\left[\phi\right] = \sum_{i=2}^{3}\left(\frac{1}{2}\bar{v}_i^2\phi_{y_i y_i} + \hat{m}^{(i)}\phi_{y_i} + \sigma v_1^{(i)}\phi_{Dy_i}\right)$$

$$+ \frac{\partial v^{1,2}}{\partial y_1}\phi_{y_2} + v^{1,2}\phi_{y_1 y_2} + \frac{\partial v^{1,3}}{\partial y_1}\phi_{y_3} + v^{1,3}\phi_{y_1 y_3} + v^{2,3}\phi_{y_2 y_3}$$

(for any test function ϕ) and L^3 and L^4 are the operators defined in appendix A (lemma A1) [with $(y_1, \bar{v}_1^2, \hat{m}^{(1)}, v_1^{(1)})$ replacing $(y, v^2, \hat{m}, v_1)$]; finally,

$$
\begin{aligned}
h_1(D, y) =\ & \hat{G}_{y_1}(D, y_1, y_2, y_3)P_D(D, y_1, y_2, y_3) - R_{y_1}(D, y_1, y_2, y_3)P(D, y_1, y_2, y_3)\\
& + M(D, y_1, y_2, y_3)
\end{aligned}
$$

$$
\begin{aligned}
h_2(D, y) =\ & \hat{G}_{y_1 y_1}(D, y_1, y_2, y_3)P_D(D, y_1, y_2, y_3) + \hat{m}^{(1)}_{y_1 y_1}(D, y_1, y_2, y_3)P_{y_1}(D, y_1, y_2, y_3)\\
& + \left[2\hat{G}_{y_1 y_1}(D, y_1, y_2, y_3) + \frac{\partial^2}{\partial y_1^2}(\sigma v_1^{(1)})(D, y_1, y_2, y_3)\right]P_{Dy_1}(D, y_1, y_2, y_3)\\
& - R_{y_1 y_1}(D, y_1, y_2, y_3)P(D, y_1, y_2, y_3) - 2R_{y_1 y_1}(D, y_1, y_2, y_3)P_{y_1}(D, y_1, y_2, y_3)\\
& + N(D, y_1, y_2, y_3)
\end{aligned}
$$

and

$$M \equiv \sum_{i=2}^{3}\left(\frac{1}{2}\frac{\partial\bar{v}_i^2}{\partial y_1}P_{y_i y_i} + \frac{\partial\hat{m}^{(i)}}{\partial y_1}P_{y_i} + \sigma\frac{\partial v_1^{(i)}}{\partial y_1}P_{Dy_i}\right) + \frac{\partial v^{2,3}}{\partial y_1}P_{y_2 y_3}$$

$$
\begin{aligned}
N \equiv\ & \sum_{i=2}^{3}\left[\frac{1}{2}\frac{\partial^2\bar{v}_i^2}{\partial y_1^2}P_{y_i y_i} + \frac{\partial\bar{v}_i^2}{\partial y_1}P_{y_1 y_i y_i} + \frac{\partial^2\hat{m}^{(i)}}{\partial y_1^2}P_{y_i} + 2\frac{\partial\hat{m}^{(i)}}{\partial y_1}P_{y_1 y_i} + \sigma\frac{\partial^2 v_1^{(i)}}{\partial y_1^2}P_{Dy_i} + 2\sigma\frac{\partial v_1^{(i)}}{\partial y_1}P_{y_1 Dy_i}\right]\\
& + \frac{\partial^2 v^{1,2}}{\partial y_1^2}P_{y_1 y_2} + \frac{\partial^2 v^{1,3}}{\partial y_1^2}P_{y_1 y_3} + \frac{\partial^2 v^{2,3}}{\partial y_1^2}P_{y_2 y_3} + 2\frac{\partial v^{2,3}}{\partial y_1}P_{y_1 y_2 y_3}
\end{aligned}
$$

38

By arguments nearly identical to the ones developed in appendix A, $u^j > 0$ (resp. < 0) whenever $h_j > 0$ (resp. < 0), $j = 1, 2$. In particular, let $\partial R / \partial D = \partial m^{(i)} / \partial D = \partial \lambda_i / \partial D = 0$, and $\sigma(D) = \sigma_0 D$, $\hat{G}(D, y) = D \cdot \hat{g}(y)$, where $\hat{g}(y) = g(y) - \sigma_0 \lambda_1(y)$, as assumed in the proposition. Then, function $P(D, y) = D \cdot p(y)$ satisfies eq. (C1), and functions h. in eq. (C2) are $h_1 = \bar{h}_1$ and $h_2 = \bar{h}_2$, where:

$$\bar{h}_1(D, y) = D \cdot \left\{ [\hat{g}_{y_1}(y) - R_{y_1}(y)] p(y) + \bar{M}(y) \right\}$$

$$\begin{aligned}\bar{h}_2(D, y) = \ & D \cdot [\hat{g}_{y_1 y_1}(y) - R_{y_1 y_1}(y)] p(y) \\ & + D \cdot \left[2\hat{g}_{y_1}(y) + \hat{m}^{(1)}_{y_1 y_1}(y) + \sigma_0 \frac{\partial^2}{\partial y_1^2} v_1^{(1)}(y) - 2R_{y_1}(y) \right] p_{y_1}(y) + D \cdot \bar{N}(y)\end{aligned}$$

and

$$\bar{M} \equiv \sum_{i=2}^{3} \left[\frac{1}{2} \frac{\partial \bar{v}_i^2}{\partial y_1} p_{y_i y_i} + \left(\frac{\partial \hat{m}^{(i)}}{\partial y_1} + \sigma_0 \frac{\partial v_1^{(i)}}{\partial y_1} \right) p_{y_i} \right] + \frac{\partial v^{2,3}}{\partial y_1} p_{y_2 y_3}$$

$$\begin{aligned}\bar{N} \equiv \ & \sum_{i=2}^{3} \left[\frac{1}{2} \frac{\partial^2 \bar{v}_i^2}{\partial y_1^2} p_{y_i y_i} + \frac{\partial \bar{v}_i^2}{\partial y_1} p_{y_1 y_i y_i} + \frac{\partial^2 \hat{m}^{(i)}}{\partial y_1^2} p_{y_i} + 2\frac{\partial \hat{m}^{(i)}}{\partial y_1} p_{y_1 y_i} + \sigma_0 \frac{\partial^2 v_1^{(i)}}{\partial y_1^2} p_{y_i} + 2\sigma_0 \frac{\partial v_1^{(i)}}{\partial y_1} p_{y_1 y_i} \right] \\ & + \frac{\partial^2 v^{1,2}}{\partial y_1^2} p_{y_1 y_2} + \frac{\partial^2 v^{1,3}}{\partial y_1^2} p_{y_1 y_3} + \frac{\partial^2 v^{2,3}}{\partial y_1^2} p_{y_2 y_3} + 2\frac{\partial v^{2,3}}{\partial y_1} p_{y_1 y_2 y_3}\end{aligned}$$

Parts a) and b) in proposition 5 then follow by arguments similar to ones used to show lemma A1: $p_{y_1} > 0$ whenever $\bar{h}_1 > 0$ and $p_{y_1 y_1} > 0$ (resp. < 0) whenever $\bar{h}_2 > 0$ (resp. < 0). $\blacksquare$

Proof of proposition 3 [Part c]. Under the scale-invariant economy considered in the main text, the price-dividend ratio p satisfies,

$$0 = \frac{1}{2} \sum_{i=1}^{3} \bar{v}_i^2 p_{y_i y_i} + v^{1,2} p_{y_1 y_2} + v^{1,3} p_{y_1 y_3} + v^{2,3} p_{y_2 y_3} + \sum_{i=1}^{3} \left(\hat{m}^{(i)} + \sigma_0 v_1^{(i)} \right) p_{y_i} - (R - \hat{g}) p + 1.$$

By tedious computations, the second partial $f \equiv p_{y_1 y_3}$ satisfies,

$$0 = \frac{1}{2} \sum_{i=1}^{3} \bar{v}_i^2 f_{y_i y_i} + v^{1,2} f_{y_1 y_2} + v^{1,3} f_{y_1 y_3} + v^{2,3} f_{y_2 y_3}$$

$$+ \left(\bar{m}^{(1)} + \frac{\partial v^{1,3}}{\partial y_3} + \frac{1}{2} \frac{\partial \bar{v}_1^2}{\partial y_1} \right) f_{y_1} + \left(\bar{m}^{(2)} + \frac{\partial v^{1,2}}{\partial y_1} \omega_{y_2} + \frac{\partial v^{2,3}}{\partial y_3} \right) f_{y_2} + \left(\bar{m}^{(3)} + \frac{\partial v^{1,3}}{\partial y_1} + \frac{1}{2} \frac{\partial \bar{v}_3^2}{\partial y_3} \right) f_{y_3}$$

$$- \left[R - \hat{g} - \frac{\partial \bar{m}^{(1)}}{\partial y_1} - \left(\frac{\partial \bar{m}^{(3)}}{\partial y_3} + \frac{\partial^2 v^{1,3}}{\partial y_1 \partial y_3} \right) \right]$$

$$+ h^f$$

where $\bar{m}^{(i)} \equiv \hat{m}^{(i)} + \sigma_0 v_1^{(i)}$ and,

$$h^f \equiv (\hat{g}_{y_1 y_3} - R_{y_1 y_3}) p + \left(\hat{g}_{y_3} - R_{y_3} + \frac{\partial^2 \bar{m}^{(1)}}{\partial y_1 \partial y_3} \right) p_{y_1} + \left(\hat{g}_{y_1} - R_{y_1} + \frac{\partial^2 \bar{m}^{(3)}}{\partial y_1 \partial y_3} \right) p_{y_3}$$

$$+ \left(\frac{\partial \bar{m}^{(1)}}{\partial y_3} + \frac{1}{2} \frac{\partial^2 \bar{v}_1^2}{\partial y_1 \partial y_3} \right) p_{y_1 y_1} + \left(\frac{\partial \bar{m}^{(3)}}{\partial y_1} + \frac{1}{2} \frac{\partial^2 \bar{v}_3^2}{\partial y_1 \partial y_3} \right) p_{y_3 y_3} + \bar{H}$$

$$\bar{H} \equiv \frac{\partial^2 \bar{m}^{(2)}}{\partial y_1 \partial y_3} p_{y_2} + \left(\frac{\partial \bar{m}^{(2)}}{\partial y_3} + \frac{\partial^2 v^{1,2}}{\partial y_1 \partial y_3} \right) p_{y_2 y_1} + \left(\frac{\partial \bar{m}^{(2)}}{\partial y_1} + \frac{\partial^2 v^{2,3}}{\partial y_1 \partial y_3} \right) p_{y_2 y_3} + \frac{1}{2} \frac{\partial^2 \bar{v}_2^2}{\partial y_1 \partial y_3} p_{y_2 y_2}$$

$$+ \frac{1}{2} \left(\sum_{i=1}^{2} \frac{\partial \bar{v}_i^2}{\partial y_3} p_{y_1 y_i y_i} + \sum_{i=2}^{3} \frac{\partial \bar{v}_i^2}{\partial y_1} p_{y_i y_i y_3} \right) + \frac{\partial v^{2,3}}{\partial y_1} p_{y_2 y_3 y_3} + \frac{\partial v^{1,2}}{\partial y_3} p_{y_1 y_1 y_2}.$$

The result follows by the assumptions in the proposition, and the Feynman-Kac representation theorem. ∎

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FOOTNOTES

[1] In the economies I consider here, these effects are direct because price-dividend ratios do not feed back risk-adjusted discount rates.

[2] See, e.g., Campbell and Shiller (1988) (approximations); Mehra and Prescott (1985), Abel (1994) or Cecchetti, Lam and Mark (1993)] (simplifying assumptions); and Campbell and Cochrane (1999), Veronesi (1999) or Chan and Kogan (2002) (numerical methods). Kogan and Uppal (2001) have developed a refined approximation approach based on asymptotic analysis ideas.

[3] Under regularity conditions, eq. (3) follows by Girsanov's change-of-measure theorem, and it can be understood as the probabilistic Feynman-Kac representation of the solution to a certain partial differential equation [namely, eq. (A1) in appendix A]. See, for example, Huang and Pagès (1992) (thm. 3, p. 53) or Wang (1993) (lemma 1, p. 202), for a series of regularity conditions underlying the Feynman-Kac theorem in infinite horizon settings arising in financial applications; and Huang and Pagès (1992) (prop. 1, p. 41) for mild regularity conditions ensuring that Girsanov's theorem holds in infinite horizon settings.

[4] Mele (2002) (appendices A, B, C) develops regularity conditions ensuring the feasibility of such a representation for a technically related problem.

[5] In their empirical work, Barsky and De Long considered feeding a variant of the Gordon's model (1962) with a (time-varying) estimate of long-term dividend growth. Naturally, the Gordon's model is based on the assumption that the dividend's growth is constant. Nevertheless, the Barsky and De Long procedure is of great interest. It highlights the role played by a convex function in vehicling small changes in expected dividend growth to large changes of the price-dividend ratios.

[6] The formal structure of the Markov chain in the two models is slightly different. In Veronesi's (1999) model (10), θ switches from the good state $\bar{\theta}$ to the bad state $\underline{\theta}$ with probability $\pi_1 dt$ (resp. θ switches from the bad state $\underline{\theta}$ to the good state $\bar{\theta}$ with probability $\pi_2 dt$) over any infinitesimal amount of time, and $k = \pi_1 + \pi_2$, $\bar{y} = \pi\bar{\theta} + (1 - \pi)\underline{\theta}$, $\pi = \pi_2 / (\pi_1 + \pi_2)$. In a simplified version of Veronesi's (2000) model, there is a probability κdt that over any infinitesimal amount of time dt, new values of θ in (11) are drawn ($\bar{\theta}$ with probability l and $\underline{\theta}$ with probability $1 - l$, and $\underline{\theta} < \bar{\theta}$), and $y^* = l\bar{\theta} + (1 - l)\underline{\theta}$.

[7] Precisely, $v_1 \equiv v_1(\gamma_*) = (\sigma_1 + \frac{1}{\sigma_0}\gamma_*)^2$, where γ_* is the positive solution to $v_1(\gamma) = \sigma_1^2 + \sigma_2^2 - 2k\gamma$. The number γ^* has the economic interpetation of variance of the prior. The results on γ^* in this example follow by theorem 12.1 in Liptser and Shiryaev (2001) (Vol. II, p. 22). They generalize results in Gennotte (1986) and are a special case of results in Detemple (1986). Gennotte and Detemple did not emphasize the impact of learning on the pricing function.

44

8 The literature on continuous time models with incomplete but *symmetric* information and Bayesian learning mechanisms is vast. It was initiated by Detemple (1986) and Gennotte (1986). David (1997) proposed the first model with unobservable processes living on a countable number of states. Veronesi (1999, 2000) and Brennan and Xia (2001) developed the first models analyzing the pricing function implications of learning phenomena.

9 By eqs. (5), returns volatility decreases with y whenever the sensitivities $\omega_i(y) \equiv p'(y) \cdot v_i(y)$ are positive and decreasing in y. In turn, ω_i $(i = 1, 2)$ decrease with y for sufficiently high levels of y whenever v_i are: a) decreasing in y; and b) negligible as y is high enough. This feature of fundamentals' volatility arises precisely in economies with learning mechanisms such as those in example 1 - in which the two sources of expected growth volatility v_i are inverse-U shaped. Intuitively, these models predict that uncertainty in expected dividend growth is almost zeroed when dividend growth is very high. As a result, asset returns volatility inherits such an extreme behavior of fundamentals' volatility.

10 See, also, Veronesi (2000, lemma 3a) for a related result.

11 More formally we have $\hat{m}''(y) = \gamma\sigma_0 v_1''(y) = 2\gamma\sigma_0 > 0$.

12 The dynamics of the expected dividend growth in eq. (16) were introduced in the term-structure literature by Ahn and Gao (1999) to describe the short-term rate dynamics.

13 Naturally, a more sensible model for applied work may feature incomplete markets or more generally, additional deviations from the Modigliani-Miller-Ricardo doctrine.

14 To produce these results, I used the following parameter values: $g_0 = 0.02$, $\sigma_0 = 0.03$, $\beta = 0.01$, $\sigma_1 = 0.15$, $\bar{y} = 0.3$, $\underline{b} = 0.10$, $\bar{b} = 0.50$, impatience rate $\delta = 0.04$, and CRRA $\eta = 2$. The values of β and σ_1 were chosen to match the typical persistence and volatility in data related to industrialized countries (*Total* Government spending over GDP on NIPA tables and OECD data). In such European countries as France, Germany, Italy, and the UK, the Gvt-ratio is currently fluctuating around 40%-50%.

15 Such an additional change of measure arises because D_t and y_t are correlated, and it is justified by the same arguments leading to eq. (13) in section 3.1. And again, $\hat{W}_2$ is a Brownian motion under the risk-neutral measure Q.

16 It is emphasized that the proofs of parts a) and b) in proposition 3 below do not rely on this scale invariance assumption. Furthermore, the restrictions stated in proposition 3 cover fewer cases than the ones allowed for by the general proofs in appendix C.