Interest Rate Variance Swaps and the Pricing of Fixed Income Volatility

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One of the pillars supporting the recent movement toward standardized measurement and trading of interest rate volatility (see http://www.garp.org/risk-news-and-resources/2013/december/a-push-to-standardize-interest-rate-volatility-trading.aspx) is a novel theory of options-based model-free fixed income volatility pricing. The meaning of this mouthful is best understood by working backwards: "fixed income volatility pricing" refers to the valuation of a contract with payoffs tied to a specific measure of realized variance of an underlying fixed income instrument (generically, a "variance swap"); "model-free" signifies the absence of reliance on modeling assumptions beyond specification of standard price dynamics and absence of arbitrage; and "options-based" relates to the valuation technique of spanning variance swap payoffs with those of options on the same underlying.

The price of volatility derived in this framework carries a clean and intuitive interpretation as the fair market value of pure "strikeless" volatility of the underlying instrument, which naturally lends itself as the basis of a benchmark index for measuring and tracking volatility for the corresponding fixed income market and serves as the underlying for standardized futures and options contracts for volatility trading.

A model-free options-based volatility pricing methodology for perpetual price processes (e.g. equity indexes, commodity prices and exchange rates) was branded and popularized as the "VIX Methodology" a decade ago by Chicago Board Options Exchange (CBOE) with an application to S&P 500 index options to create the VIX Index. Since then, the same methodology has been carried over to other markets, such as those for gold, oil and single stocks.

In contrast, to create analogous volatility indexes for the complex and varied world of fixed income markets, one must first relax the original assumption of constant interest rates, and explicitly account for the distinct characteristics of each market. An early illustration of this can be seen in CBOE's design of its first fixed income volatility index (the interest rate swap volatility index, or SRVX), which measures swaption-implied volatility of forward swap rates. This article provides an overview of how volatility pricing and indexing methodologies work for options on four major fixed income contracts: government bond futures (Mele and Obayashi, aka MO, 2013a); time deposit futures (MO 2013b); interest rate swaps (MO 2012); and credit default swap (CDS) indexes (MO 2013c).

Government Bonds

Pricing volatility in government bond markets is deceptively challenging, mainly because of the high dimensionality of these markets. Suppose we wish to price the one-month forward volatility of a three-month future on the 10-year Treasury note, and that available for trading are American-style options expiring in one month. Can these options be used to price the desired volatility in a strictly model-free fashion?

The practical answer depends on the magnitude of two issues. One relates to the early exercise premium embedded in American option prices. A second relates to the mismatch in maturity between the options and the underlying futures — i.e., one-month options are used to span risks generated by three-month futures returns. This second source is specific to the nature of fixed income markets.

Fortunately, in practice, situations arise in which the model-dependent components may be presumed small enough, such that the model-free component of the formula is a sufficient approximation of the true price of volatility. MO (2013d, Chap-

ter 4) provide further details on how one may estimate these approximation errors. For example, a numerical experiment based on Vasicek (1977) model of short-term interest rate dynamics under realistic US Treasury market conditions indicates that (1) the early exercise premium embedded in American options does not inflate the European option-based index by more than one relative percentage point; and (2) the impact of the maturity mismatch is negligible when the difference in maturities is as small as two months, as in the example above.

Maturity and Numéraires Mismatches

To illustrate the maturity mismatch issue, suppose that available for trading are European-style options expiring at T on a 10-year Treasury note forward expiring at $S \geq T$. The option span operates under the so-called T-forward probability, whereas the forward price is a martingale only under the S-forward probability, which generates risks in the forward price that cannot be spanned by the set of all available options. Unless $T=S$, we are pricing the forward volatility with the "wrong" probability.

This issue is reminiscent of convexity problems arising in fixed income security evaluation (see, e.g., Brigo and Mercurio, 2006; Veronesi, 2010). In our case, model-free indexes in the spirit of the standard VIX Methodology cannot be calculated at the strictest level of theoretical rigor.

The silver lining is that, as long as T and S-T are small, the numerical impact of both the maturity mismatch and early exercise premium is likely to be small enough to be ignored, as is done in the implementation of the CBOE/CBOT 10-Year Treasury Note Volatility Index (VXTYN). Still, it is important to realize that a VIX-like implementation leads to an approximation of the true fair value of a variance swap, and to know when the approximation is of a tolerable magnitude given the context.

Let $F_t(S,T)$ be the forward price at t, for delivery at S, of a coupon bearing bond expiring at T, with $t \leq S \leq T$, i.e. $F_t(S,T) = \frac{B_t(\mathbb{T})}{P_t(S)}$, where $P_t(S)$ is the price at t of a zero coupon bond expiring at $S \geq t$, and $B_t(T)$ is the price at t of the underlying bond.

Let r_t be the instantaneous short-term rate process and let Q be the risk-neutral probability. It is well-known (see, e.g., Mele, 2013, Chapter 12) that in a diffusion setting, $F_t(S,T)$ satisfies

$$\frac{dF_\tau(S,\mathbb{T})}{F_\tau(S,\mathbb{T})} = v_\tau(S,\mathbb{T}) \cdot dW_{F^S}(\tau), \quad \tau \in (t,S), \tag{1}$$

where $v_\tau(S,\mathbb{T})$ is the instantaneous volatility process adapted to $W_{F^S}(\tau)$, a multidimensional Brownian motion under the S-forward probability, Q_{F^S}, defined through the Radon-Nikodym derivative, as follows:

$$\left.\frac{dQ_{F^S}}{dQ}\right|_{\mathcal{G}_S} = \frac{e^{-\int_t^S r_\tau d\tau}}{P_t(S)}, \tag{2}$$

and $\mathcal{G}_S$ denotes the information set at time S.

Consider the following payoff of a variance swap:

$$\pi(T,\mathbb{T}) \equiv V_t(T,S,\mathbb{T}) - \mathbb{P}(t,T,S,\mathbb{T}), \quad T \leq S,$$

where $V_t(T,S,\mathbb{T}) \equiv \int_t^T \|v_\tau(S,\mathbb{T})\|^2 d\tau$ is the percentage integrated variance, and the fair value of the strike $\mathbb{P}(t,T,S,\mathbb{T})$ is

$$\mathbb{P}(t,T,S,\mathbb{T}) = \frac{1}{P_t(T)}\mathbb{E}_t\left[e^{-\int_t^T r_\tau d\tau} V_t(T,S,\mathbb{T})\right] = \mathbb{E}_t^{Q_{F^T}}(V_t(T,S,\mathbb{T})), \tag{3}$$

where the second equality follows by a change of probability, $\mathbb{E}_t$ denotes conditional expectation under the risk-neutral probability, and $\mathbb{E}_t^{Q_{F^T}}$ under the T-forward probability. That is, the fair value of the variance swap is the expected realized variance under the T-forward probability. It is a notable point of departure from the standard equity case in which the fair value of a variance swap is the risk-neutral expectation, assuming interest rates are constant.

For the case at hand, we obviously cannot assume constant interest rates. Moreover, we still need to evaluate the RHS of Eq. (3). Following the VIX Methodology, one might try to link the expectation in Eq. (3) to the value of a log-contract (Neuberger, 1994) — i.e., a contract with a payoff equal to $\ln \frac{F_T(S,\mathbb{T})}{F_t(S,\mathbb{T})}$ at time T. In the standard equity case, the value of a log-contract is indeed minus one half the expected realized variance. In our case, it follows by Eq. (1) and Itô's lemma that

$$\mathbb{E}_t^{Q_{F^S}}(V_t(T,S,\mathbb{T})) = -2\mathbb{E}_t^{Q_{F^S}}\left(\ln \frac{F_T(S,\mathbb{T})}{F_t(S,\mathbb{T})}\right). \tag{4}$$

However, Eq. (4) does not link to Eq. (3). The expectation in Eq. (3) is the fair value of a variance contract expiring at T, whereas the expectation on the LHS of Eq. (4) is the fair value of a variance contract expiring at S. In other words, the market numéraire for government bond contracts expiring at a given maturity $\mathcal{Y}$ is the price of zeros expiring at $\mathcal{Y}$, where $\mathcal{Y} = T$ in Eq. (3) and $\mathcal{Y} = S$ in Eq. (4).

To evaluate the RHS of Eq. (3), we need something more than Itô's lemma. We have to acknowledge that the dynamics of $F_t(S,T)$ under the "variance swap" pricing measure are those

under Q_{F^T}, not under Q_{F^S}, as in Eq. (1). Now, by Girsanov's theorem, we have the following:

$$\frac{dF_\tau(S,\mathbb{T})}{F_\tau(S,\mathbb{T})} = -v_\tau(S,\mathbb{T})\left(v_\tau(T,\mathbb{T}) - v_\tau(S,\mathbb{T})\right)d\tau + v_\tau(S,\mathbb{T})\cdot dW_{F^T}(\tau), \quad \tau \in (t,T), \tag{5}$$

where $W_{F^T}(\tau)$ is a multidimensional Brownian motion under Q_{F^T}. That is, the forward price is not a martingale under Q_{F^T}, such that by Itô's lemma,

$$\mathbb{E}_t^{Q_{F^T}}\left(V_t(T,S,\mathbb{T})\right) = 2\mathbb{E}_t^{Q_{F^T}}\left(\tilde{\ell}(t,T,S,\mathbb{T})\right) - 2\mathbb{E}_t^{Q_{F^T}}\left(\ln\frac{F_T(S,\mathbb{T})}{F_t(S,\mathbb{T})}\right), \tag{6}$$

where we have defined

$$\tilde{\ell}(t,T,S,\mathbb{T}) \equiv -\int_t^T v_\tau(S,\mathbb{T})\left(v_\tau(T,\mathbb{T}) - v_\tau(S,\mathbb{T})\right)d\tau. \tag{7}$$

The two expressions in Eqs. (4) and (6) differ by the " tilting" term $\tilde{\ell}(t,T,S,\mathbb{T})$ in addition to the fact that the expectations on the RHS are taken under two different probabilities. The tilting term encapsulates the impact of the maturity mismatch: the forward price is a martingale under Q_{F^S} and yet we are insisting in evaluating the expectation of its realized variance under Q_{F^T}, motivated as we are in our search for the fair value of the original variance swap$\mathbb{P}(t,T,S,\mathbb{T})$ in Eq. (3). We expect $\tilde{\ell}(t,T,S,\mathbb{T})$ to be zero only when T=S.

Spanning

We are not done yet, as we still need to derive the value of the log-contract in Eq. (6). It is natural at this juncture to rely on the " spanning" approach, led by Demeterfi, Derman, Kamal and Zou (1999); Bakshi and Madan (2000); Britten-Jones and Neuberger (2000); and Carr and Madan (2001), among others. This approach links the value of the log- (and other) contracts to that of a portfolio of out-of-the-money (OTM) European-style options.

To apply these spanning arguments to our context, consider a Taylor's expansion with remainder

$$\ln\frac{F_T(S,\mathbb{T})}{F_t(S,\mathbb{T})} = \frac{F_T(S,\mathbb{T}) - F_t(S,\mathbb{T})}{F_t(S,\mathbb{T})} - \left(\int_0^{F_t(S,\mathbb{T})}(K - F_T(S,\mathbb{T}))^+\frac{1}{K^2}dK + \int_{F_t(S,\mathbb{T})}^\infty (F_T(S,\mathbb{T}) - K)^+\frac{1}{K^2}dK\right),$$

and take expectations under Q_{F^T},

$$\mathbb{E}_t^{Q_{F^T}}\left(\ln\frac{F_T(S,\mathbb{T})}{F_t(S,\mathbb{T})}\right) = \frac{\mathbb{E}_t^{Q_{F^T}}\left(F_T(S,\mathbb{T})\right)}{F_t(S,\mathbb{T})} - 1 - \frac{1}{P_t(T)}\left(\int_0^{F_t(S,\mathbb{T})}\mathrm{Put}_t(K)\frac{1}{K^2}dK + \int_{F_t(S,\mathbb{T})}^\infty \mathrm{Call}_t(K)\frac{1}{K^2}dK\right), \tag{8}$$

where $\mathrm{Put}_t(K)$ and $\mathrm{Call}_t(K)$ denote the prices of European puts and calls struck at K, and we have relied on the following pricing equations:

$$\frac{\mathrm{Put}_t(K)}{P_t(T)} = \mathbb{E}_t^{Q_{F^T}}\left((K - F_T(S,\mathbb{T}))^+\right), \quad \frac{\mathrm{Call}_t(K)}{P_t(T)} = \mathbb{E}_t^{Q_{F^T}}\left((F_T(S,\mathbb{T}) - K)^+\right).$$

Because $F_t(S,\mathbb{T})$ is not a martingale under Q_{F^T}, the first term on the RHS of Eq. (8) is not one, but equals $\mathbb{E}_t^{Q_{F^T}}\left(e^{\tilde{\ell}(t,T,S,\mathbb{T})}\right)$, where $\tilde{\ell}(t,T,S,\mathbb{T})$ is as in Eq. (7). Utilizing this expression and combining Eq. (8) with Eq. (6), we find that the fair value of the variance swap in Eq. (3) is,

$$\mathbb{P}(t,T,S,\mathbb{T}) = 2\left(1 - \mathbb{E}_t^{Q_{F^T}}\left(e^{\tilde{\ell}(t,T,S,\mathbb{T})} - \tilde{\ell}(t,T,S,\mathbb{T})\right)\right) + \frac{2}{P_t(T)}\left(\int_0^{F_t(S,\mathbb{T})}\mathrm{Put}_t(K)\frac{1}{K^2}dK + \int_{F_t(S,\mathbb{T})}^\infty \mathrm{Call}_t(K)\frac{1}{K^2}dK\right). \tag{9}$$

Accordingly, a formulation of an index of government bond volatility is

$$\boxed{\text{GB-VI}(t,T,S,\mathbb{T}) \equiv \sqrt{\frac{1}{T-t}\mathbb{P}(t,T,S,\mathbb{T})}}$$

where $\mathbb{P}(t,T,S,\mathbb{T})$ is as in Eq. (9).

Time Deposits

Time deposit variance contracts share an interesting feature with government bonds: they can be priced based on the same change-of-numéraire set forth in the previous section. A point of departure arises when expressing time deposit volatility in terms of *basis point volatility of rates*, as opposed to the more familiar notion of percentage volatility of prices. To accommodate this practice, we need to consider spanning arguments different from those in the previous section.

The Underlying Risks

Let $l_t(\Delta)$ be the simply compounded interest rate on a deposit for the time period from t to t+Δ. As a non-limitative illustration, we refer to $l_t(\Delta)$ as the LIBOR.

Define a forward contract as one where at time t, one party agrees to pay a counterparty a payoff equal to $100\times(1-l_s(\Delta)) - Z_t(S,S+\Delta)$ at time S. The forward LIBOR price, $Z_t(S,S+\Delta)$, is agreed at time t such that in the absence of arbitrage

$$Z_t(S,S+\Delta) = 100\times\left(1 - f_t(S,S+\Delta)\right), \tag{10}$$

where $f_t(S,S+\Delta)$ is the forward LIBOR, which satisfies: $f_t(S,S+\Delta)$=$\mathbb{E}_t^{Q_{F^S}}(l_S(\Delta))$. Because $l_s(\Delta)$=$f_s(S,S+\Delta)$, $f_t(S,S+\Delta)$ is a

martingale under Q_{F^S}. Therefore, assuming that the information in this market is driven by Brownian motions, the forward price, $Z_t(S,S+\Delta)$, satisfies the following:

$$\frac{dZ_\tau(S,S+\Delta)}{Z_\tau(S,S+\Delta)} = v_\tau^z(S,\Delta)\,dW_{F^S}(\tau), \quad \tau \in (t,S), \tag{11}$$

where $W_{F^S}(\tau)$ is a multidimensional Brownian motion under Q_{F^S}, and $v_\tau^z(S,\Delta)$ is a vector of instantaneous volatilities, adapted to $W_{F^S}(\tau)$.

Because we shall deal with variance swap security designs referencing rates instead of prices, we consider the forward LIBOR equivalent to Eq. (11), as follows:

$$\frac{df_\tau(S,S+\Delta)}{f_\tau(S,S+\Delta)} = v_\tau^f(S,\Delta)\,dW_{F^S}(\tau), \quad \tau \in (t,S), \tag{12}$$

where, by Itô's lemma,

$$v_\tau^f(S,\Delta) \equiv \left(1 - f_\tau^{-1}(S,S+\Delta)\right) v_\tau^z(S,\Delta).$$

LIBOR Variance Contracts and Volatility Indexes

We define the basis point LIBOR integrated *rate*-variance as,

$$V_t^{f,\mathrm{bp}}(T,S,\Delta) \equiv \int_t^T f_\tau^2(S,S+\Delta)\left\|v_\tau^f(S,\Delta)\right\|^2 d\tau,$$

such that, by arguments similar to those leading to Eq. (3) in the previous section, the fair value of the time deposit rate-variance swap generated at t, and paying off at T, is the following:

$$V_t^{f,\mathrm{bp}}(T,S,\Delta) - \mathbb{P}_f^{\mathrm{bp}}(t,T,S,\Delta), \quad T \le S,$$

is

$$\mathbb{P}_f^{\mathrm{bp}}(t,T,S,\Delta) = \mathbb{E}_t^{Q_{F^T}}\left(V_t^{f,\mathrm{bp}}(T,S,\Delta)\right). \tag{13}$$

We face two complications. The first is the same maturity mismatch arising in the government bond case. The second complication is that we are dealing with a notion of basis point variance, which necessitates a different treatment from the percentage case.

In the percentage case, it is by now well-understood that its formulation relates to a log-contract. The contract we shall link the expectation on the RHS of Eq. (13) is, instead, a "quadratic" contract delivering a payoff equal to $f_T^2(S,S+\Delta) - f_t^2(S,S+\Delta)$. To see how this contract is useful, note that by Itô's lemma and the Girsanov theorem,

$$\mathbb{E}_t^{Q_{F^T}}\left(f_T^2(S,S+\Delta)\right) - f_t^2(S,S+\Delta) = 2\mathbb{E}_t^{Q_{F^T}}\left(\tilde{\ell}^{\mathrm{bp}}(t,T,S)\right) + \mathbb{P}_f^{\mathrm{bp}}(t,T,S,\Delta), \tag{14}$$

where

$$\tilde{\ell}^{\mathrm{bp}}(t,T,S) \equiv \int_t^T f_\tau^2(S,S+\Delta)\,v_\tau^f(S,\Delta)\left(v_\tau^f(T,\Delta) - v_\tau^f(S,\Delta)\right)d\tau. \tag{15}$$

On the other hand, by taking expectations under the T-forward probability of a Taylor's expansion of $f_T^2(S,S+\Delta)$ with remainder, we obtain the following:

$$\begin{aligned}&\mathbb{E}_t^{Q_{F^T}}\left(f_T^2(S,S+\Delta)\right) - f_t^2(S,S+\Delta)\\ &= 2f_t^2(S,S+\Delta)\left(\mathbb{E}_t^{Q_{F^T}}\left(e^{\tilde{\ell}_f(t,T,S)}\right) - 1\right)\\ &+ \frac{2}{P_t(T)}\left(\int_0^{f_t(S,S+\Delta)} \mathrm{Put}_t^f(K_f,T,S,\Delta)\,dK_f + \int_{f_t(S,S+\Delta)}^\infty \mathrm{Call}_t^f(K_f,T,S,\Delta)\,dK_f\right),\end{aligned} \tag{16}$$

where,

$$\tilde{\ell}_f(t,T,S) \equiv \int_t^T v_\tau^f(S,\Delta)\left(v_\tau^f(T,\Delta) - v_\tau^f(S,\Delta)\right)d\tau, \tag{17}$$

and:

$$\mathrm{Put}_t^f(K_f,T,\Delta) = \frac{\mathrm{Call}_t^z(100(1-K_f),T,S,\Delta)}{100}, \quad \mathrm{Call}_t^f(K_f,T,\Delta) = \frac{\mathrm{Put}_t^z(100(1-K_f),T,S,\Delta)}{100}.$$

$\mathrm{Put}_t^z(K_z,T,S,\Delta)$ and $\mathrm{Call}_t^z(K_z,T,S,\Delta)$ are the prices of OTM puts and calls written on $Z_t(S,S+\Delta)$ with strike price K_z and maturity T; $\mathrm{Put}_t^z(K_z,T,S,\Delta)$ and $\mathrm{Call}_t^z(K_z,T,S,\Delta)$ are hypothetical OTM European-style options on the forward LIBOR rate.

Finally, by matching Eq. (16) to Eq. (14), we are able to use options on Z to price the basis point volatility of f, as follows:

$$\begin{aligned}\mathbb{P}_f^{\mathrm{bp}}(t,T,S,\Delta) &= 2\left(f_t^2(S,S+\Delta)\left(\mathbb{E}_t^{Q_{F^T}}\left(e^{\tilde{\ell}_f(t,T,S)}\right) - 1\right) - \mathbb{E}_t^{Q_{F^T}}\left(\tilde{\ell}^{\mathrm{bp}}(t,T,S)\right)\right)\\ &+ \frac{2}{P_t(T)}\left(\int_0^{f_t(S,S+\Delta)} \mathrm{Put}_t^f(K_f,T,S,\Delta)\,dK_f + \int_{f_t(S,S+\Delta)}^\infty \mathrm{Call}_t^f(K_f,T,S,\Delta)\,dK_f\right)\end{aligned} \tag{18}$$

where $\tilde{\ell}_f(t,T,S)$ and $\tilde{\ell}^{\mathrm{bp}}(t,T,S)$ are defined in Eqs. (17) and (15).

An index of basis point time deposit rate-volatility is,

$$\boxed{\text{TD-VI}_f^{\mathrm{bp}}(t,T,S,\Delta) \equiv 100^2 \times \sqrt{\frac{\mathbb{P}_f^{\mathrm{bp}}(t,T,S,\Delta)}{T-t}}}$$

where $\mathbb{P}_f^{\mathrm{bp}}(t,T,S,\Delta)$ is as in Eq. (18).

Note that the above formula involves an equally-weighted portfolio of OTM options. This is in contrast to the percentage volatility case where the weight of each option is inversely proportional to the square of its strike. MO (2012; 2013d) provide additional intuition about this feature by showing how the hedging portfolio of a basis point volatility index differs from that of a percentage index, and provide further analytical details on the behavior of the two indexes.

MO (2012) contains the first formulation of a variance con-

tract design cast in basis point terms that applies to fixed income markets–namely, to interest rate swaps. In earlier work, Carr and Corso (2001) explain how to hedge the variance of price changes in markets with constant interest rates. MO (2013d, Chapter 2) explain that the elegant replication arguments in Carr and Corso (2001) break down once interest rates are random, but that the random numéraire inherent in each market of interest in the fixed income space can be incorporated into the replicating portfolios and the variance contract design.

The next section illustrates how a variance swap design is affected by the numéraire when deriving model-free indexes in interest rate swap markets.

Interest Rate Swaps

How is the variance swap security design affected by the presence of more complex numéraires than the price of a zero? We now consider the pricing of variance swaps in the context of interest rate swap markets. For the sake of brevity, we focus on the basis point variance derivation to mirror market practice, which is also the formulation used for the CBOE SRVX Index.

Let $R_t(T_1,\cdots,T_n)$ be the forward swap rate prevailing at t, – i.e., the fixed rate such that the value of a forward starting swap (at $T=T_0$, with reset dates $T_0,\cdots,T_{n-1}$, tenor length T_n-T, and payment periods $T_1-T_1,\cdots,T_n-T_{n-1}$) is zero at t. It is well-known (e.g., Mele, 2013, Chapter 12) that $R_t(T_1,\cdots,T_n)$ is a martingale under the so-called annuity probability Q_A defined through the Radon-Nikodym derivative, as follows:

$$\left.\frac{dQ_{\mathrm{A}}}{dQ}\right|_{\mathcal{G}_T} = e^{-\int_t^T r_s ds}\frac{\mathrm{PVBP}_T(T_1,\cdots,T_n)}{\mathrm{PVBP}_t(T_1,\cdots,T_n)},$$

and $\mathrm{PVBP}_t(T_1,\cdots,T_n)$ is the "price value of the basis point" i.e., the value at t of annuity paid over the swap tenor.

We assume that $R_t(T_1,\cdots,T_n)$ is a diffusion process, as follows:

$$dR_\tau(T_1,\cdots,T_n) = R_\tau(T_1,\cdots,T_n)\,\sigma_\tau(T_1,\cdots,T_n)\cdot dW_{\mathrm{A}}(\tau), \quad \tau\in[t,T], \tag{19}$$

where $W_A(\tau)$ is a Brownian Motion under Q_A, and $\sigma_\tau(T_1,\cdots,T_n)$ is adapted to $W_A(\tau)$, and define the basis point realized variance of the forward swap rate arithmetic changes in the time interval $[t,T]$,[1]

$$V_n^{\mathrm{bp}}(t,T) \equiv \int_t^T R_\tau^2(T_1,\cdots,T_n)\,\|\sigma_\tau(T_1,\cdots,T_n)\|^2\,d\tau.$$

How do we design a variance contract in this case that allows model-free valuation? Consider the value of a payer swap with fixed rate, K:

$$\mathrm{Swap}_T(K;T_1,\cdots,T_n) \equiv \mathrm{PVBP}_T(T_1,\cdots,T_n)\left[R_T(T_1,\cdots,T_n)-K\right].$$

The payoff of a payer swaption is $\max\{\mathrm{Swap}_T(K;\cdot),0\}$ and that of a receiver is $\max\{-\mathrm{Swap}_T(K;\cdot),0\}$. Notice that swaption prices contain information about both interest rate volatility and the value of an annuity. This reveals yet another fundamental difference between equity and fixed income markets. Options on equities relate to a single source of risk: the stock price. Instead, swaps and swaptions are affected by two sources of risk: the swap rate, $R_T(\cdot)$, and the annuity factor, $\mathrm{PVBP}_T(\cdot)$.

MO (2012) show how to insulate the pure interest rate volatility component in a model-free fashion. They show that the annuity factor entering into the payoff of swaptions needs to be worked into the variance swap design to allow for model-free pricing of the variance contract in both percentage and basis point variance terms. More generally, MO (2013d, Chapter 2) develop a framework that handles any market and numéraire of interest in the fixed income space. Let us illustrate how this framework specializes in the interest rate swap case.

Consider a variance swap starting at t with the following payoff:

$$\pi_n(t,T) \equiv \left(V_n^{\mathrm{bp}}(t,T)-\mathbb{P}_n(t,T)\right)\times \mathrm{PVBP}_T(T_1,\cdots,T_n)$$

at T, where $\mathbb{P}_n(t,T)$ is the fair value of the contract such that

$$\mathbb{P}_n(t,T) = \mathbb{E}_t^{\mathrm{A}}\left(V_n^{\mathrm{bp}}(t,T)\right), \tag{20}$$

where E_t^A denotes conditional expectation under the annuity probability Q_A. Moreover, by Eq. (19) and Itô's lemma,

$$\mathbb{E}_t^{\mathrm{A}}\left(R_T^2(T_1,\cdots,T_n)\right)-R_t^2(T_1,\cdots,T_n) = \mathbb{E}_t^{\mathrm{A}}\left(V_n^{\mathrm{bp}}(t,T)\right) = \mathbb{P}_n(t,T), \tag{21}$$

where the second equality follows by Eq. (20).

Similar to our derivation of Eq. (16), we now take the expectation under the annuity probability Q_A of a Taylor's expansion of $R_T^2(T_1,\cdots,T_n)$ with remainder, obtaining:

$$\begin{aligned}&\mathbb{E}_t^{Q_{FT}}\left(R_T^2(T_1,\cdots,T_n)\right)-R_t^2(T_1,\cdots,T_n) \\ &= \frac{2}{\mathrm{PVBP}_t(T_1,\cdots,T_n)}\left(\int_0^{R_t(T_1,\cdots,T_n)}\mathrm{Swpn}_{n,t}^r(K,T)\,dK+\int_{R_t(T_1,\cdots,T_n)}^{\infty}\mathrm{Swpn}_{n,t}^p(K,T)\,dK\right),\end{aligned} \tag{22}$$

where $\mathrm{Swpn}^r_{n,t}(K,T)$ and $\mathrm{Swpn}^p_{n,t}(K,T)$ denote the prices of receiver and payer swaptions at t, referencing a swap with tenor T_n-T and strike K, and expiring at T.

Matching Eq. (21) to Eq. (22) leaves:

$$\mathbb{P}_n(t,T) = \frac{2}{\mathrm{PVBP}_t(T_1,\cdots,T_n)}\left(\int_0^{R_t(T_1,\cdots,T_n)} \mathrm{Swpn}^r_{n,t}(K,T)\,dK + \int_{R_t(T_1,\cdots,T_n)}^{\infty} \mathrm{Swpn}^p_{n,t}(K,T)\,dK\right). \tag{23}$$

Eq. (23) provides the expression for the value of the variance swap in a model-free fashion. It is a portfolio of equally weighted swaptions, as in the case for time deposits (see Eq. 18); however, this portfolio is rescaled by the inverse of the annuity factor (the numéraire in the interest rate swap market), whereas the portfolio in (18) is rescaled by the price of a zero (the numéraire in the government bond market).[2]

Intuitively, tilting a variance swap by the market numéraire at T (which is one in the cases dealt with in sections on time deposits and interest rate swaps) causes its fair value to be defined under a market space where all the relevant information is given by the price of available derivatives — an expectation under the numéraire probability of interest.[3]

Finally, an index of interest rate swap volatility is

$$\boxed{\text{IRS-VI}_n^{\text{bp}}(t,T) \equiv 100^2 \times \sqrt{\frac{1}{T-t}\mathbb{P}_n(t,T)}}$$

where $\mathbb{P}_n(t,T)$ is as in Eq. (23).

Credit

Credit volatility can be priced through credit default swaptions, although the nature of credit risk calls for a number of new features to take into account. For example, we need to consider credit variance swaps on loss-adjusted forward position in a CDS index, and we must deal with the survival contingent probability and the defaultable annuity market numéraire to account for default risk.

We only present the percentage variance contract formulation here. The risk we are dealing with is that of a CDS index, for which a buyer pays periodic premium (the CDS index spread) and the seller insures losses from defaults by any of the index's constituents during the term of the contract. If a constituent defaults, the defaulted obligor is removed from the index, and the index continues to be traded with a prorated notional amount. Options on a CDS index are European-style, to buy (payers) or sell (receivers) protection at the strike spread upon option expiry.

We assume credit events may occur over a sequence of regular intervals (T_{i-1}, T_i) with length 1/b, for $i=1,\cdots,bM$, where M is the number of years the index runs, T_0 is the time of the index origination. We assume that (1) loss-given-default (LGD) is constant; (2) the short-term rate r_τ is a diffusion process; and (3) default arrives as a Cox process with intensity λ adapted to r. Let n be the initial number of names in the index decided at time $t \equiv T_0$, and let each constituent have a notional value 1/n, the same LGD, and the same intensity, λ.

The number of names having survived up to T_i is $\mathcal{S}(T_i) \equiv \sum_{j=1}^{n}(1-\mathbb{I}_{\{\tau_j \le T_i\}})$ where τ_j is the time at which name j defaults, and the outstanding notional is $\mathcal{N}(\tau)=1/n\,\mathcal{S}(\tau)$ with $\mathcal{N}(t)\equiv 1$. The index loss at τ_j should obligor j default is $\mathrm{LGD}\frac{1}{n}\mathbb{I}_{\{t\le\tau_j\le T_{bM}\}}$, whereas the premium at T_i is $\frac{1}{b}\overline{\mathrm{CDX}}_t(M)\times\frac{1}{n}\mathcal{S}(T_i)$. Finally, the value of protection leg minus premium leg is

$$\mathrm{DSX}_t = \mathrm{LGD}\cdot v_{0t} - \frac{1}{b}\overline{\mathrm{CDX}}_t(M)\cdot v_{1t},$$

where v_{0t} is the value at t of \$1 paid off at the time of default of a representative firm, provided default occurs before the index expiry; and where v_{1t} is the value at t of an annuity of \$1 paid at $T_1,\cdots,T_{bM}$, until either a default of the representative firm or the expiry of the index (whichever occurs first) the value of a *defaultable annuity*.

A CDS index payer is an option to enter a CDS index at T as a protection buyer with strike spread K. Upon exercise, the protection buyer would also receive a front-end protection arising from losses occurring before the option matures. Accordingly, consider a loss-adjusted forward position at t in a CDS index that starts at T, which can be shown to equal the following:

$$\mathrm{DSX}^L_{t,T}(\tau) \equiv \frac{1}{b}\mathcal{N}(\tau)\,v_{1\tau}\left(\mathrm{CDX}_\tau(M) - \overline{\mathrm{CDX}}_t(M)\right),$$

where $\mathrm{CDX}_\tau(M)$ is defined as the value of $\mathrm{CDX}_\tau(M)$, set such that a forward position at τ in the index is worthless, viz $\mathrm{DSX}^L_{\tau,T}(\tau)=0$,

$$\frac{1}{b}\mathrm{CDX}_\tau(M) = \mathrm{LGD}\frac{v_{0\tau}}{v_{1\tau}} + \frac{v^{\mathrm{F}}_\tau}{\mathcal{N}(\tau)\,v_{1\tau}}.$$

Note that $\mathcal{N}(\tau)\,v_{1\tau}$ is the natural numéraire in this market. Indeed, $\mathrm{CDX}_t(M)$ is a martingale under the "survival contingent probability" Q_{sc}, defined through the Radon-Nikodym derivative, as follows:

$$\left.\frac{dQ_{\mathrm{sc}}}{dQ}\right|_{\mathbb{F}^r_T} = e^{-\int_\tau^T r(u)\,du}\frac{\mathcal{N}(T)\,v_{1T}}{\mathcal{N}(\tau)\,v_{1\tau}},$$

where $\mathbb{F}^r_T$ denotes the information set at time T, which includes the path of the short-term rate only. The prices of a

payer and receiver with strike K expiring at T, are, for any $\tau \in [t, T]$, $SW^{p}_{\tau}(K,T;M) \equiv N(\tau)\, v_{1\tau} \cdot E^{sc}_{\tau}[(CDX_T(M)-K)^{+}]$ and $SW^{r}_{\tau}(K,T;M) \equiv N(\tau)\, v_{1\tau} \cdot E^{sc}_{\tau}[K - (CDX_T(M))^{+}]$.

We assume that

$$\frac{d\mathrm{CDX}_\tau(M)}{\mathrm{CDX}_\tau(M)} = -\left(\mathbb{E}^{\mathrm{sc}}_\tau\left(e^{j(\tau;M)}-1\right)\eta(\tau)\right)d\tau + \sigma(\tau;M)\cdot dW^{\mathrm{sc}}(\tau) + \left(e^{j(\tau;M)}-1\right)dJ^{\mathrm{sc}}(\tau),$$

where $W^{sc}(\tau)$ is a multidimensional Brownian motion and $J^{sc}(\tau)$ is a Cox process (with intensity $\eta(\cdot)$ and jump size $j(\cdot)$) under the survival contingent probability.

We wish to price a credit variance swap originated at t, and paying off at T, as follows:

$$\left(V_M(t,T) - \mathbb{P}_{\mathrm{var},M}(t,T)\right) \times \mathcal{N}(T)\, v_{1T},$$

where $\mathbb{P}_{\mathrm{var,M}}(t,T)$ is the fair value of the contract, and we have defined the percentage variance as,

$$V_M(t,T) \equiv \int_t^T \|\sigma(\tau;M)\|^2 d\tau + \int_t^T j^2(\tau;M)\, dJ^{\mathrm{sc}}(\tau).$$

In MO (2013d, Chapter 5, Appendix D), we show that

$$\mathbb{P}_{\mathrm{var},M}(t,T) = \frac{2}{v_{1t}}\left[\int_0^{\mathrm{CDX}_t(M)} \frac{\mathrm{SW}^r_t(K,T;M)}{K^2}dK + \int_{\mathrm{CDX}_t(M)}^{\infty} \frac{\mathrm{SW}^p_t(K,T;M)}{K^2}dK\right] - 2\mathbb{E}^{\mathrm{sc}}_t\left[\int_t^T\left(e^{j(\tau;M)}-1-j(\tau;M)-\frac{1}{2}j^2(\tau;M)\right)dJ^{\mathrm{sc}}(\tau)\right].$$

Accordingly, a credit volatility index is

$$\boxed{\text{C-VI}_M(t,T) \equiv 100 \times \sqrt{\frac{1}{T-t}\mathbb{P}_{\mathrm{var},M}(t,T)}}$$

where $\mathbb{P}_{\mathrm{var,M}}(t,T)$ is as in Eq. (24).

The first term on the RHS of Eq. (24) is model-free, once we estimate the CDX default intensity incorporated by v_{1t}. The second term is small for all intents and purposes, and should not materially affect the value of the index approximated by only retaining the first term.

Closing Thoughts

The viability of volatility indexing using the methodologies described in this article depends in part on the unique microstructure of each fixed income market considered, which varies across time and geographies. A sufficiently high quality of price discovery, breadth of strikes and depth of liquidity in the options used in the various index formulas render the resulting indexes viable for serving as the underlying of tradable products such as volatility futures and options.

The CBOE has already launched a pair of indexes — the VXTYN and SRVX (covering near-month CBOT options on 10-year T-Note futures and one-year/10-year USD swaptions, respectively) — that possess these qualities. Moreover, additional applications may come in other geographies and fixed income asset classes.

From a methodological research perspective, the logical next step in supporting the creation of a market for standardized fixed income volatility trading is to model prices of futures and options on these indexes in a way that is consistent with the underlying yield or credit curves, the indexes themselves and the term structure of volatility in order to facilitate risk management and formulation of trading strategies by end users.

FOOTNOTES

1. MO (2012, 2013d) considered a framework in which Eq. (19) is extended to cover the presence of jumps, and showed the remarkable property that the fair value of a variance swap remains the same in this case.

2. Merener (2012) considers the replication of variance contracts on swap rates by relying on a flat forward curve approximation, and does not rescale for the relevant notion of market numéraire in the manner we suggest in this section, thereby providing neither the index formulae and pricing in this section nor the hedging details described in MO (2012; 2013, Chapter 3). Rheinlaender (2013) considers variance swaps priced and hedged based on parametric assumptions.

3. Interestingly, our model-free expression for the variance swap in Eq. (23) coincides with the conditional second moment of the forward swap rate, as it turns out by comparing our formula in Eq. (23) with expressions in Trolle and Schwartz (2011). It is an interesting statistical property that complements the asset pricing foundations laid down in this section and in MO (2012; 2013, Chapter 3). MO (2013, Chapter 2) and Mele, Obayashi and Shalen (2013) describe the behavior of variance risk-premiums under both interpretations of $\mathbb{P}_n(t,T)$ in Eq. (23).

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