



Asymmetric stock market volatility and the cyclical behavior of expected returns[☆]

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Abstract

Recent explanations of aggregate stock market fluctuations suggest that countercyclical stock market volatility is consistent with rational asset evaluations. In this paper, I develop a framework to study the causes of countercyclical stock market volatility. I find that countercyclical risk premia do not imply countercyclical return volatility. Instead, countercyclical stock volatility occurs if risk premia increase more in bad times than they decrease in good times, thereby inducing price–dividend ratios to fluctuate more in bad times than in good. The business cycle asymmetry in the investors' attitude toward discounting future cash flows plays a novel and critical role in many rational explanations of asset price fluctuations.

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1. Introduction

Why does stock market volatility vary over time? Economists have been intrigued by this issue for decades. For example, Schwert (1989b) finds that the volatility of no single macroeconomic variable could help explain low frequency movements of aggregate stock market volatility. Yet stock market volatility is related to the business cycle. A number of empirical studies confirm further findings from Schwert (1989a, b) that the volatility of stock returns is higher in bad times than in good times (see, e.g., Brandt and Kang, 2004, and the additional evidence provided here). This paper addresses an important but still unanswered question: Why is stock market volatility asymmetric over the business cycle?

My central result is that, in economies with rational expectations, return volatility is countercyclical because risk premia (i.e., the compensation investors require to invest in the stock market) change asymmetrically in response to variations in economic conditions. That risk premia are countercyclical has been a widely known empirical fact since the seminal contributions of Fama and French (1989) and Ferson and Harvey (1991). However, the main message of this paper is not a simple statement that risk premia must be countercyclical to generate countercyclical return volatility. Instead, the crucial point is that, to induce countercyclical return volatility, risk premia must increase more in bad times than they decrease in good times, a new hypothesis that I support with substantial empirical evidence.

So why do asymmetric risk premia fluctuations translate into countercyclical return volatility? Consider Fig. 1, in which I assume that the investors' risk-adjusted discount rates are inversely and asymmetrically related to some variable y that tracks the state of the economy. This asymmetry implies that in good times investors do not significantly alter the discount rates used to evaluate future dividends. Consequently, price–dividend ratios do not fluctuate widely in good times. In bad times, however, the investors' discount rates are extremely sensitive to changes in economic conditions. Therefore, variations in the price–dividend ratios become increasingly volatile as economic conditions deteriorate. The main result of this paper is that these asymmetric movements of the price–dividend ratios occur when the asymmetry in discounting is sufficiently pronounced. I calculate a theoretical lower bound for the asymmetric movements of the risk premia that triggers the

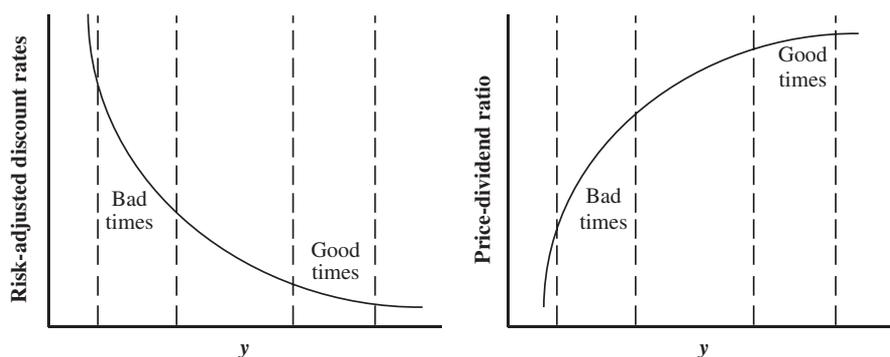


Fig. 1. Countercyclical return volatility. If price–dividend ratios are concave in some state variable y tracking the state of the economy, then return volatility increases on the downside and is consequently countercyclical. According to the theory in this article, price–dividend ratios are concave in y if the risk-adjusted discount rates are decreasing and sufficiently convex in y .

previous asymmetric variations in the price–dividend ratios. This bound can be tight. For example, economies exist in which risk premia are countercyclical but do not satisfy this bound and, consequently, induce price–dividend ratios to fluctuate more in good times than in bad.

Naturally, countercyclical return volatility could also arise because the volatility of the state variables in the economy is inherently countercyclical. Alternatively, the conditions developed here highlight the mechanism through which countercyclical return volatility is endogenously induced by rational fluctuations of the price–dividend ratio. Moreover, empirical evidence suggests that price–dividend ratios exhibit the pattern predicted in this paper. I find that, over the last 50 years, price–dividend ratios movements in the US have been asymmetric over the business cycle: Downward changes occurring in recessions have been far more severe than upward changes during expansions.

In the economy I study, dividend growth is independent and identically distributed, while interest rates and risk premia are driven by a state variable that is interpreted as an index of the state of the economy. This economy is rich enough to include many model examples in the literature. The distinctive feature of this article is the way I deal with interest rates and risk premia. The standard approach is to link interest rates and risk premia to markets, preferences, and technology (e.g., Basak and Cuoco, 1998; Campbell and Cochrane, 1999; Jermann, 2005) or in general to make use of higher level assumptions about the exact relations among interest rates, risk premia, and the primitives of the economy (e.g., Brennan, Wang, and Xia, 2004; Lettau and Wachter, 2007).

In this paper, I take an opposite approach. Instead of making assumptions on interest rates and risk premia, I look for pricing kernels that make return volatility countercyclical. It is this search process that leads to the predictions summarized in Fig. 1. One additional contribution of the paper is to use these new predictions to understand when, why, and how models with time-varying discount rates could predict countercyclical volatility. For example, in a seminal contribution Campbell and Cochrane (1999) find that models with external habit formation might lead to countercyclical volatility. This paper explains the rationale behind this important result. At the same time, the predictions developed here go well beyond the case of habit formation.

Countercyclical stock volatility is an empirical observation related to the so-called feedback effect; i.e., the effect by which asset returns and return volatility are negatively correlated. Indeed, this paper shows that a pronounced asymmetric behavior of the risk premia leads return volatility to be higher in bad times (when ex post returns are low) than in good (when ex post returns are high). Moreover, the asymmetric behavior of the risk premia could help explain why return volatility increases after prices fall. According to the explanations summarized in Fig. 1, return volatility increases after a price drop, i.e., when the price–dividend ratios enter the volatile region in Fig. 1.

Campbell and Hentschel (1992) develop the first partial equilibrium explanation for the feedback effect. But, their explanation relies on a different channel. In the Campbell and Hentschel economy, the negative correlation between return volatility and returns arises through the combination of two inextricable effects: first, risk premia rise (and hence prices fall) with the volatility of dividend news; second, return volatility increases with the volatility of dividend news. Thus, in the Campbell and Hentschel economy the feedback effect arises because there is fluctuating economic uncertainty (i.e., dividend volatility is random) and investors fear this uncertainty.

Wu (2001), Bansal and Yaron (2004), and Tauchen (2005) reconsider this channel of fluctuating economic uncertainty. Bansal and Yaron as well as Tauchen show that, in general equilibrium, investors with a preference for early resolution of uncertainty require compensation for economic uncertainty, thereby inducing negative co-movements between ex post returns and return volatility. This explanation of the feedback effect is not inconsistent with my explanation based on an asymmetric behavior of the risk premia. In fact, the last contribution of this paper is an extension of my previous analysis to economies in which the fundamentals are surrounded by fluctuating uncertainty.

I consider two sources of volatility for the fundamentals of the economy. One is related to uncertain consumption growth volatility while the other, suggested by Tauchen (2005), relates to higher order uncertainty about consumption growth (the volatility of volatility). I show when and how the risk premia for these sources of uncertainty make prices fall after a rise in economic uncertainty. I provide a new role for the price–dividend ratio. For example, the relation between prices and the volatility of volatility is not uniquely tied down by the level of the volatility risk premia. It also depends on how asymmetrically the price–dividend ratio reacts to changes in consumption growth volatility. Moreover, I show that if investors have a preference for early resolution of uncertainty, an increase in the economic uncertainty can lower the risk-free rate, thereby producing a positive relation between asset prices and economic uncertainty. In particular, I show that the feedback effect arises when the volatility of volatility is not too responsive to changes in volatility, thereby dampening the effects associated with the preference for early resolution of uncertainty. I use these novel insights to shed new light on previous models of fluctuating economic uncertainty.

The main scope of this paper is to isolate the business cycle determinants of return volatility. Its focus is on channels of asymmetric volatility that are markedly distinct from the leverage effects (the effects by which an increase in the debt-to-equity ratio boosts firms' volatility). Instead, the general equilibrium analysis of leverage effects is in Aydemir, Gallmeyer, and Hollifield (2005), who conclude that these effects have marginal quantitative implications at the market level.

The paper is organized as follows. In Section 2, I develop the core analysis. Section 3 hinges upon this analysis and provides examples of economies with countercyclical stock volatility. It also contains a calibration experiment to illustrate the key quantitative implications of the paper. Section 4 develops extensions and identifies conditions under which fluctuating economic uncertainty induces asset returns and volatility to co-move negatively. Section 5 concludes. The appendix contains technical details and proofs.

2. The dynamics of volatility under general no-arbitrage restrictions

This section develops the main result of the paper. In Section 2.1, I describe the economic environment. In Sections 2.2 and 2.3, I present and discuss general test conditions under which the volatility of asset returns is countercyclical.

2.1. The economy

I consider a pure exchange, frictionless economy endowed with a single consumption good. Let the process $\{D_t\}_{t \geq 0}$ be the instantaneous rate of consumption. I assume that consumption equals the dividends paid by a long-lived asset. Accordingly, the terms

“consumption” and “dividends” are used interchangeably. Let $\{y_t\}_{t \geq 0}$ be an additional state variable. I assume that (D_t, y_t) forms a diffusion process. A long-lived asset is an asset that promises to pay $\{D_t\}_{t \geq 0}$. Let $\{P_t\}_{t \geq 0}$ be the corresponding asset price process. As is well known, the absence of arbitrage opportunities implies that there exists a positive pricing kernel $\{\xi_t\}_{t \geq 0}$ such that

$$P_t \xi_t = E_t \left[\int_t^\infty \xi_s D_s ds \right], \quad t \geq 0, \tag{1}$$

where $E_t[\cdot]$ denotes the expectation operator conditional on the information available at time t .¹ Bubbles are not considered in this paper. Moreover, I assume that the total consumption endowment D_t is generated by a geometric Brownian motion

$$\frac{dD_t}{D_t} = g_0 dt + \sigma_0 dW_{1t}, \tag{2}$$

where W_1 is a standard Brownian motion and g_0 and σ_0 are positive constants. Finally, the state variable y_t is a stationary process. It solves

$$dy_t = m(y_t) dt + v_1(y_t) dW_{1t} + v_2(y_t) dW_{2t}, \tag{3}$$

where W_2 is another independent standard Brownian motion, and $m, v_1,$ and v_2 ($v_i > 0, i = 1, 2$) are given functions that guarantee a strong solution to the previous equation. It is well known (e.g., Duffie, 2001) that in this environment the pricing kernel ξ_t in Eq. (1) is the solution to

$$\frac{d\xi_t}{\xi_t} = -R_t dt - \lambda_{1t} dW_{1t} - \lambda_{2t} dW_{2t}, \quad \xi_0 = 1, \tag{4}$$

for some processes R_t and λ_{it} . The economic interpretation of R and λ_i is also standard. In Eq. (4), R is the instantaneous interest rate and $\lambda = [\lambda_1 \lambda_2]^T$ is the vector of unit prices of risk related to the sources of risk W_1 and W_2 . I now formulate the main assumption in this paper.

Assumption 1 (*Scale-invariant economies*). The instantaneous interest rate, R , and the unit prices of risk, λ_i , are functions of the state variable y only. That is, $R_t = R(y_t)$ and $\lambda_{it} = \lambda_i(y_t)$, where the functions $R(y), \lambda_1(y),$ and $\lambda_2(y)$ are twice continuously differentiable.

Assumption 1 guarantees that the price–dividend ratio P_t/D_t is a function p of the state variable y_t only,

$$\frac{P_t}{D_t} = p(y_t), \tag{5}$$

whence the scale-invariant terminology. In many existing models, y_t is a variable related to the general state of the economy, i.e., an expansion state variable summarizing the business cycle conditions (see Section 3). This is also the interpretation of y_t here. Accordingly, I refer to the state variable y_t as the state of the economy and to any variable positively (negatively) correlated with y_t as procyclical (countercyclical).

¹The conditional expectation in Eq. (1) is taken with respect to the filtration generated by the Brownian motions driving the dynamics of (D_t, y_t) [see Eqs. (2) and (3) below], augmented by the null sets. Proposition 1 shows that, under Assumption 1, the relevant conditioning information at time t is the state (D_t, y_t) at time t .

Which properties of the price–dividend ratio p are sought in this model? We want that the price–dividend ratio p reacts asymmetrically to changes in y_t . Precisely, we want that the price–dividend ratio decreases more in bad times (when y_t is low) than it increases in good times (when y_t is high). A key observation is that, to satisfy this property, the price–dividend ratio must be increasing and concave in y_t , as Fig. 1 suggests.

To formalize this intuition, let $p'(y) \equiv (d/dy)p(y)$ and $p''(y) \equiv (d^2/dy^2)p(y)$ denote the first and second order derivatives of the price–dividend ratio with respect to y , respectively, and consider the pricing equation $0 = \xi_t D_t dt + E_t[d(\xi_t P_t)]$. Under regularity conditions,² an application of Itô’s lemma to the definition of returns $(dP_t + D_t dt)/P_t$ yields

$$\text{Returns}_t = \mathcal{E}(y_t) dt + \text{Vol}_1(y_t) dW_{1t} + \text{Vol}_2(y_t) dW_{2t}, \tag{6}$$

where

$$\mathcal{E}(y) = \text{Expected returns} = R(y) + \beta_{CF} \lambda_1(y) + \beta_{P/D}(y) \cdot \lambda(y) \quad \text{and} \tag{7}$$

$$\text{Vol}(y) = [\text{Vol}_1(y) \text{Vol}_2(y)] = \text{Return volatility} = [\beta_{CF} + \beta_{P/D,1}(y) \beta_{P/D,2}(y)], \tag{8}$$

and the two components of $\beta_{P/D}$ (i.e., $\beta_{P/D,1}$ and $\beta_{P/D,2}$) and β_{CF} are given by

$$\beta_{P/D,i}(y) = v_i(y) \frac{p'(y)}{p(y)}, \quad i = 1, 2 \quad \text{and} \tag{9}$$

$$\beta_{CF} = \sigma_0. \tag{10}$$

For reasons developed below, it is also important to analyze the determinants of the price–dividend ratio volatility. By another application of Itô’s lemma,

$$dp(y_t) = E_t[dp(y_t)] + \text{Vol}_1^p(y_t) dW_{1t} + \text{Vol}_2^p(y_t) dW_{2t}, \tag{11}$$

where

$$\text{Vol}^p(y) = [\text{Vol}_1^p(y) \text{Vol}_2^p(y)] = \text{Price–dividend ratio volatility} = [v_1(y) v_2(y)] \cdot p'(y). \tag{12}$$

As is clear, return volatility in Eq. (8) is affected by the two volatility components $v_i(y_t)$ of y_t (which are exogenous) and by the term $p'(y_t)/p(y_t)$ (which is endogenous). The endogenous term $p'(y_t)/p(y_t)$ is the price-induced component of return volatility. It is decreasing in y_t , and thus countercyclical, whenever the price–dividend ratio p is increasing and concave in y_t , i.e., $p'(y_t) > 0$ and $p''(y_t) \leq 0$. The intuition here is that asymmetric fluctuations in the price–dividend ratio induce return volatility to increase on the downside. However, because the price–dividend ratio is endogenous, not all possible primitives in the economy lead to this asymmetry. We therefore need to figure out the right primitives (m, v_1 , and v_2), interest rates, and risk premia that do create the desired asymmetric pattern.

Definition 1 summarizes the key features needed from the price–dividend ratio to induce asymmetric movements in return volatility.

²The basic regularity conditions in this paper are that the price–dividend ratio p is twice differentiable and that it and its derivatives admit the Feynman and Kac representation. See Mele (2003, 2005) for technical details and references related to the feasibility of these conditions in both finite and infinite horizon settings.

Definition 1 (*Asymmetric return volatility*). Return volatility is asymmetric if the two volatility components of $\text{Vol}(y) = [\text{Vol}_1(y) \text{Vol}_2(y)]$ in Eq. (8) are countercyclical, i.e., if $\text{Vol}_i(y)$ is decreasing in y for $i = 1, 2$. Moreover,

- (a) The price-induced component of return volatility in Eq. (8) is asymmetric if the price-elasticity $E_p(y) \equiv p'(y)/p(y)$ is countercyclical, i.e., if $E_p(y)$ is decreasing in y .
- (b) The price-induced component of the price-dividend ratio volatility in Eq. (12) is asymmetric if the price-sensitivity $S_p(y) \equiv p'(y)$ is countercyclical, i.e. if $S_p(y)$ is decreasing in y .

Eq. (8) shows that return volatility is asymmetric if the price-dividend betas $\beta_{P/D,i}(y_t) = (p'(y_t)/p(y_t))v_i(y_t)$ are countercyclical. This property occurs if the volatilities of $y_t, v_i(y_t)$, and the price-elasticity $E_p(y_t)$ are both countercyclical. Because the volatilities $v_i(y_t)$ are exogenous, they play a relatively straightforward role in this paper. The more ambitious purpose here is to focus on channels of asymmetric volatility arising through no-arbitrage, countercyclical movements of the price-elasticity and the price-sensitivity.³

The condition that the price-sensitivity $S_p(y_t)$ be countercyclical is important for at least two reasons. First, there is strong evidence in the US that the price-dividend ratios decrease more in bad times (during recessions) than they increase in good times (during expansions) (see Section 3.3). A satisfactory explanation of asymmetric volatility must be consistent with this important empirical regularity. Second, the empirical evidence in this paper suggests that variations in the level of the price-dividend ratios display countercyclical volatility in the US. Both of these empirical regularities can be made consistent with rational asset evaluation if the price-sensitivity $S_p(y_t)$ is countercyclical.

2.2. The cyclical properties of price-dividend ratios and return volatility

I now proceed to state the main result of the paper. To prepare the discussion of this result, it is useful to introduce two fundamental concepts. First, define

$$\bar{m}(y_t) = m(y_t) - \sum_{i=1}^2 \lambda_i(y_t)v_i(y_t) + \sigma_0v_1(y_t). \tag{13}$$

In short, $\bar{m}(y_t)$ equals the risk-adjusted drift $\frac{d}{dt} \mathbb{E}_t(y_\tau)|_{\tau=t} = m(y_t) - \sum_{i=1}^2 \lambda_i(y_t)v_i(y_t)$, plus the instantaneous covariance between dividend growth $\frac{dD_t}{D_t}$ and changes in the state dy_t , i.e., $\text{cov}_t(\frac{dD_t}{D_t}, dy_t) \equiv \frac{d}{dt} \mathbb{E}_t[(\frac{D_\tau - D_t}{D_t})(y_\tau - y_t)]|_{\tau=t} = \sigma_0v_1(y_t)$. The conditional expectation $\mathbb{E}_t(\cdot)$ is taken under the risk-neutral probability, defined through the risk-neutral evaluation equation $\mathbb{E}_t(dP_t) + D_t dt = R_t P_t dt$. Note that I am not assuming that the risk-neutral probability is unique in the model. Instead, I am looking for pricing kernels (and, hence, risk-neutral probabilities) that make return volatility asymmetric.

The second important definition relates to the expected returns in Eq. (7), which I decompose as

$$\mathcal{E}(y) = \text{Disc}(y) + \beta_{P/D}(y) \cdot \lambda(y), \tag{14}$$

³The price-elasticity $E_p(y)$ in Definition 1a parallels the notion of relative basis risk for a discount bond introduced by Cox, Ingersoll, and Ross (1979, p. 56). I refer to $E_p(y)$ as “elasticity,” not “semi-elasticity,” to simplify the exposition.

where

$$\text{Disc}(y) = R(y) + \beta_{CF}\lambda_1(y) \tag{15}$$

are the discount rates adjusted for cash flow risk (concisely, the risk-adjusted discount rates) and $\beta_{P/D}(y) \cdot \lambda(y)$ is the additional compensation for the risk of fluctuations in the price–dividend ratio. In this model, both $\lambda_1(y)$ and the cash flow beta β_{CF} are exogenous. Therefore, the discount rates $\text{Disc}(y)$ are also exogenous. In contrast, the vector of price–dividend betas $\beta_{P/D}(y)$ is endogenous as it depends on the properties of the price–dividend ratio.

Proposition 1 isolates the key properties of the price–dividend ratio along with their implications on the volatility components in Definition 1.

Proposition 1. *Let the endowment process be as in Eq. (2), interest rates and unit-risk premia be as in Assumption 1, and $\{y_t\}_{t \geq 0}$ be the solution to Eq. (3). Then, the price P_t is such that $P_t = D_t \cdot p(y_t)$, where p is a positive function satisfying the following properties:*

- (a) *Suppose that the risk-adjusted discount rates $\text{Disc}(y)$ are countercyclical, i.e., $\frac{d}{dy}\text{Disc}(y) < 0$. Then, the price–dividend ratio is procyclical, i.e., $\frac{d}{dy}p(y) > 0$. Moreover, suppose that*
 - (a.1) $\frac{d^2}{dy^2}\text{Disc}(y) > 0$ (Asymmetric discount rates), and
 - (a.2) $\mathcal{A}(y) \equiv \frac{d^2}{dy^2}\bar{m}(y) - 2\frac{d}{dy}\text{Disc}(y) < 0$ (Asymmetric expectations). Then, the price–dividend ratio reacts asymmetrically to variations in the state of the economy, i.e., it is concave in y : $\frac{d^2}{dy^2}p(y) < 0$. Consequently, (i) the price-induced components of volatility in Definition 1 are asymmetric and (ii) the return volatility $\text{Vol}(y)$ in Eq. (8) is countercyclical for all values of the state y on which the volatilities of the state $v_i(y)$ are not increasing.
- (b) *Suppose that $y \in (\underline{y}, \bar{y})$ for two constants \underline{y} and \bar{y} , and assume that one of the two following conditions holds true: (i) $\lim_{y \rightarrow \underline{y}} \text{Disc}(y) = \infty$ (Large discounting) or (ii) the risk-adjusted discount rates $\text{Disc}(y)$ are bounded and decreasing in y for all $y \in (\underline{y}, \bar{y})$, but $\lim_{y \rightarrow \underline{y}} \frac{d}{dy}\text{Disc}(y) = -\infty$ (Large asymmetry in discounting). Then, under the technical regularity conditions in the appendix (conditions H1), there exists a threshold level of the state $y^* > \underline{y}$ such that the conclusions of the previous part hold true for all $y \in (\underline{y}, y^*)$.*

Conversely, suppose that the discount rates $\text{Disc}(y)$ are countercyclical and that the price-induced components of volatility in Definition 1 are asymmetric. Then, either Condition a.1 or Condition a.2, or both, hold on some range of the state y having strictly positive probability.

Proposition 1 identifies necessary and sufficient conditions leading to asymmetric volatility through countercyclical movements of the price-elasticity $E_p(y_t)$ and the price-sensitivity $S_p(y_t)$. In Section 3.3, I produce a calibration experiment to assess the extent to which these conditions are consistent with the empirical evidence on risk premia and the price–dividend ratios in the US. I now develop the economic interpretation of the conditions in Proposition 1.

2.3. Discussion

Proposition 1 imposes joint restrictions on the price–dividend ratio, the law of motion for the state variable y_t (i.e., on m , v_1 , and v_2), and the risk-adjusted discount rates $\text{Disc}(y_t)$ introduced in Eq. (15). The first restriction in Part (a) formalizes a well known concept. If risk-adjusted discount rates are countercyclical, price–dividend ratios are procyclical. For example, suppose that investors become more risk-averse during recessions. Then, in bad times investors discount future cash flows more heavily, thereby driving price–dividend ratios down.

Proposition 1a isolates the conditions under which the price–dividend ratio reacts asymmetrically to changes in the state of the economy. It imposes two basic conditions. The first condition, a.1, requires that the discount rates increase more in bad times than they decrease in good times. The economic intuition underlying this condition has been developed in the Introduction (see Fig. 1). While somewhat technical, the second condition, a.2, is also economically important. Consider the evaluation formula in Eq. (1). Under the risk-neutral probability,

$$\begin{aligned} \frac{P_0}{D} &= \mathbb{E} \left[\int_0^\infty e^{-\int_0^t R(y_u) du} \frac{D_t}{D} dt \middle| y_0 = y \right] \\ &= \mathbb{E} \left[\int_0^\infty e^{-\int_0^t R(y_u) du} \cdot e^{(g_0 - \frac{1}{2}\sigma_0^2)t - \int_0^t \sigma_0 \lambda_1(y_u) du + \sigma_0 \hat{W}_{1,t}} dt \middle| y_0 = y \right], \end{aligned} \tag{16}$$

where the second equality follows by applying Itô’s lemma to Eq. (2), \hat{W}_1 is a standard Brownian motion under the risk-neutral probability Q (say), and the risk-adjustment term $\int_0^t \sigma_0 \lambda_1(y_u) du$ arises as the conditional expectation $\mathbb{E}[\cdot | y_0 = y]$ is taken under the risk-neutral probability. By replacing the definition of $\text{Disc}(y) \equiv R(y) + \sigma_0 \lambda_1(y)$ in Eq. (15) into the previous equation, we can rewrite the price–dividend ratio $\frac{P_0}{D} = p(y)$ as follows,

$$p(y) = \mathbb{E} \left[\int_0^\infty \frac{D_t^*}{D} \cdot e^{-\int_0^t \text{Disc}(y_u) du} dt \middle| y_0 = y \right], \quad \frac{D_t^*}{D} \equiv e^{(g_0 - \frac{1}{2}\sigma_0^2)t + \sigma_0 \hat{W}_{1,t}}. \tag{17}$$

Eq. (17) is a present value formula in which a fictitious risk-unadjusted dividend growth $\frac{D^*}{D}$ is discounted using the risk-adjusted rates $\text{Disc}(y)$. According to Eq. (17), changes in prices reflect the investors’ risk-adjusted expectation about the future state of the economy and, hence, the discount rates to prevail in the future. In addition, the future state of the economy is correlated with dividend growth. Therefore, changes in prices should also factor in the covariance between dividend growth and changes in the state of the economy, $\text{cov}_t(\frac{dD_t}{D_t}, dy_t)$.

Condition a.2 in Proposition 1a formalizes the intuition that the price–dividend ratio in Eq. (17) is affected by the risk-neutral expectation of the state and the covariance between dividend growth and changes in the state. Recall the definition of $\mathcal{A}(y)$ in Condition a.2,

$$\mathcal{A}(y) = \frac{d^2}{dy^2} \bar{m}(y) - 2 \frac{d}{dy} \text{Disc}(y). \tag{18}$$

By definition, $\bar{m}(y_t)$ is the sum of the risk-adjusted expectation of the instantaneous changes in y_t , $\frac{d}{dt} \mathbb{E}_t(y_t) |_{\tau=t}$ and the covariance $\text{cov}_t(\frac{dD_t}{D_t}, dy_t)$ [see Eq. (13)]. Thus, $\bar{m}(y)$ summarizes the expectation about changes in the state of the economy and the movements of the state with dividend growth. The function $\bar{m}(y)$ can also be interpreted as

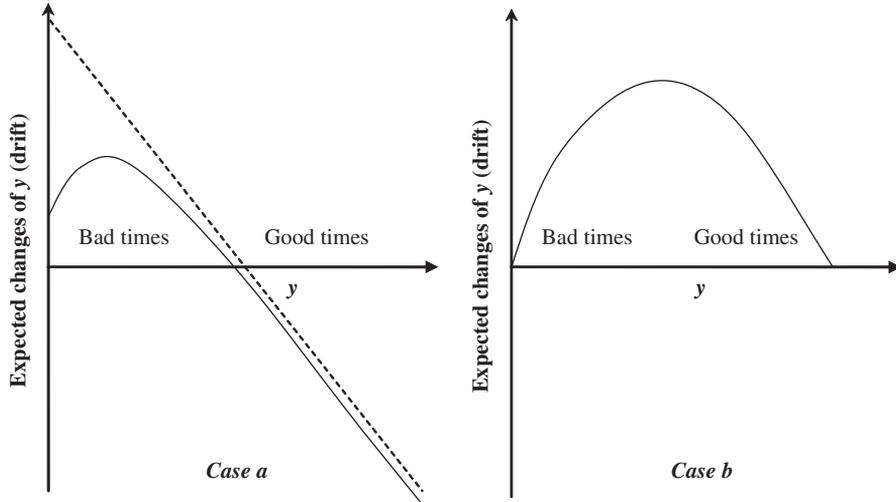


Fig. 2. Expected changes in the state of the economy. In this picture, y is a state variable tracking the state of the economy. The left-hand panel (Case a) depicts the drift function $\bar{m}(y_t) = \frac{d}{dt} \bar{E}_t(y_t)|_{\tau=t}$ that makes the volatility of the expected changes in y more volatile in bad times than in good times (solid line). The expectation \bar{E} is taken under the probability \bar{Q} in Eq. (19). The dashed line depicts one example of the expectation $\frac{d}{dt} E_t(y_t)|_{\tau=t}$ taken under the physical probability. The right-hand panel (Case b) depicts the drift function $\bar{m}(y_t)$ that makes the volatility of the expected changes in y as volatile in good as in bad times.

follows. Rewrite the expectation in Eq. (17) as

$$p(y) = \bar{E} \left[\int_0^\infty e^{g_0 t - \int_0^t \text{Disc}(y_u) du} dt \middle| y_0 = y \right], \tag{19}$$

where $\bar{E}[\cdot]$ is the conditional expectation taken under a new probability \bar{Q} defined by the Radon-Nikodym derivative $d\bar{Q}/dQ = \exp(-\frac{1}{2}\sigma_0^2 t + \sigma_0 \hat{W}_{1t})$ [see Eq. (51) in the appendix for the derivation]. Eq. (19) is still a present value formula, in which a fictitious deterministic dividend growth $e^{g_0 t}$ is discounted using the risk-adjusted rates $\text{Disc}(y)$, under \bar{Q} . Compared with the density of y under Q , the density of y under \bar{Q} is right-shifted to reflect the positive covariance $cov_t(\frac{dD}{Dt}, dy_t)$. Then, $\bar{m}(y)$ is the drift of y under \bar{Q} , and Condition a.2 is satisfied if $\bar{m}(y)$ is sufficiently concave, i.e. $\frac{d^2}{dy^2} \bar{m}(y) < 2 \frac{d}{dy} \text{Disc}(y) < 0$, where the last inequality follows by the proposition’s assumption that the risk-adjusted discount rates are countercyclical.

What does this concavity mean economically? Fig. 2 illustrates it. In Case a, the expected changes in y_t are more volatile in bad times; in Case b, the expected changes in y_t are volatile in both bad and good times. In both cases, the investors’ expectation about their own future discount rates fluctuates more in bad times than in good times.⁴ These

⁴This statement is formally shown in the appendix (see Corollary 1). Intuitively, we have that, in Case a, the volatile expectations of y_t in bad times translate to volatile expectations of future discount rates. In contrast, in Case b, the volatile expectations of y_t in good times do not translate to volatile expectations of future discount rates. This is because, in good times, investors expect declining rates of growth in the expansion state variable y_t (see Fig. 2, Case b). Hence, they do not expect their discount rates to fall significantly in the future.

asymmetric expectation shifts amplify the asymmetric fluctuations of the current discount rates induced by Condition a.1. Therefore, they accentuate the asymmetric behavior of the price–dividend ratio, and contribute to countercyclical variation in the price-induced components of volatility in Definition 1.

These expectation asymmetries take place under the probability \bar{Q} in Eq. (19). Thus, they could arise for at least two reasons: (1) the expectation of the future state of the economy under the physical probability is inherently asymmetric; and/or (2) investors are risk-averse. To isolate the effects associated with risk-aversion, consider the extreme situation in which the expectation of the future state of the economy is not asymmetric at all, as in Case a of Fig. 2 (the dashed line). To make the same expectation asymmetric under the probability \bar{Q} (the solid line in Fig. 2, Case a), investors should require a risk-aversion correction that is more pronounced in bad times than in good. In other words, Condition a.2 imposes that the risk-aversion correction be sufficiently asymmetric to generate the drift distortion in Fig. 2. In turn, this distortion alters the strength of mean-reversion of y_t in bad times. It makes bad times more persistent than good times under the risk-neutral probability, thereby implying a slow decay rate for the price of Arrow–Debreu securities paying off in future bad states.

Finally, Part (b) of Proposition 1 deals with the extreme situation in which the discount rates are large (or change quite asymmetrically) in bad times. Intuitively, this part of the proposition follows from Eq. (17). If bad times worsen, the proposition's conditions imply that investors raise their discount rates to the extent that the price–dividend ratio collapses to very low values. As a result, the price-elasticity $E_p(y_t)$ and sensitivity $S_p(y_t)$ increase in bad times. Section 3 shows that such an extreme asymmetry in discounting can occur in economies with external habit formation or restricted stock market participation.

3. Examples

Proposition 1 provides general insights into the cyclical determinants of stock return volatility. These same insights can be used to interpret the empirical success or failure of previous existing models of aggregate stock market fluctuations. In Sections 3.1 and 3.2, I use Proposition 1 to analyze and compare economies with external habit formation and economies with restricted stock market participation. In Section 3.3, I produce a calibration experiment to illustrate the quantitative content of the theory developed in this paper.

3.1. External habit formation

Example 1 contains a well known model that can be analyzed with the tools introduced in this paper.

Example 1 (*Campbell and Cochrane, 1999*). Consider an infinite horizon economy in which a representative agent has discounted utility $e^{-\rho t}u(c, x) = e^{-\rho t} \frac{1}{1-\eta} [(c-x)^{1-\eta} - 1]$, where ρ is the subjective discount rate, c is consumption, and x is the habit stock. Let $y_t \equiv (c_t - x_t)/c_t$ be the surplus consumption ratio. By assumption, y_t is the solution to

$$dy_t = y_t[\kappa(\bar{y} - \log y_t) + \frac{1}{2}\sigma_0^2 I(y_t)^2]dt + \sigma_0 y_t I(y_t) dW_{1t}, \quad (20)$$

where $\kappa > 0, \bar{y} \in \mathbb{R}, l(y) = \frac{1}{\bar{Y}}\sqrt{1 + 2(\bar{y} - \log y)} - 1, y \in (0, \bar{Y} \cdot e^{1/2(1 - \bar{Y}^2)})$, and $\bar{Y} = \exp(\bar{y}) = \sigma_0\sqrt{\eta/\kappa}$. In equilibrium, $c_t = D_t$ for all t , the interest rate R is constant, and the unit risk premia are $\lambda_2(y) = 0$ and $\lambda_1(y) = \eta\sigma_0[1 + l(y)]$.

Campbell and Cochrane identify the habit formation mechanics that lead to pricing kernel properties in line with empirical facts. The message of their model, however, is far more general. For example, Guvenen (2005) shows that models with restricted stock market participation have a reduced form, which is similar to the habit formation model of Campbell and Cochrane. (See, also, the similarities in Section 3.2 below.)

In the Campbell and Cochrane economy, return volatility is such that $\text{Vol}_2(y_t) = 0$ and $\text{Vol}(y_t) \equiv \text{Vol}_1(y_t)$, where, by Eq. (8),

$$\text{Vol}(y_t) = \sigma_0[1 + E_p(y_t)y_t l(y_t)], \tag{21}$$

and $E_p(y_t)$ is the price-elasticity introduced in Definition 1. Campbell and Cochrane demonstrate numerically that, in a discrete time version of Example 1, the return volatility in Eq. (21) is decreasing in the surplus consumption ratio, y_t . There are two reasons for this result.

First, in the empirically relevant range of variation, the volatility of the fundamentals, $\sigma_0 y_t l(y_t)$, is decreasing in y_t . Second, the price–dividend ratio collapses to zero as the surplus consumption ratio goes to zero. The second effect arises through the channel identified by Proposition 1b. In the Campbell and Cochrane economy, the discount rates $\text{Disc}(y_t)$ react asymmetrically to changes in y_t . In particular, they become large and infinitely convex as the surplus ratio gets smaller (two properties labeled “large discounting” and “large asymmetry in discounting” in Proposition 1b). Therefore, even if the volatility of fundamentals $\sigma_0 y_t l(y_t)$ approaches zero as the surplus ratio approaches zero, the extremely high discounting in bad times makes the price–dividend ratio very small, thereby inducing the price-sensitivity $S_p(y_t)$ and the price-elasticity $E_p(y_t)$ to blow up.⁵ As a result, the return volatility, $\text{Vol}(y_t)$, is such that $\text{Vol}(y_t) > \sigma_0$ for small values of y_t .⁶

Do habit models always predict that price–dividend ratios change asymmetrically in response to variations of the surplus consumption ratio? Consider Example 2.

Example 2. Assume that, in the habit formation economy of Example 1, the representative agent has instantaneous utility $u(c_t, x_t) = \frac{1}{1-\eta}[(c_t - x_t)^{1-\eta} - 1]$ but that the surplus consumption ratio $y_t \equiv (c_t - x_t)/c_t$ is such that $G_t \equiv y_t^{-\gamma}$ is the solution to

$$dG_t = k(\bar{G} - G_t) dt - \alpha(G_t - l)\sigma_0 dW_{1t}, \tag{22}$$

for some positive constants k, \bar{G}, α , and l .

⁵As the surplus ratio y_t approaches zero, the surplus volatility $\sigma_0 y_t l(y_t)$ also approaches zero. This condition guarantees that y_t remains positive, a requirement consistent with the preferences specification underlying the habit formation mechanics.

⁶Wachter (2005) uses new numerical methods and shows in detail the concavity property that emerges by Proposition 1b. It is also possible to show that, with the parameter values used by Campbell and Cochrane, Condition a.2 in Proposition 1a is not satisfied, although the same condition is satisfied with different parameter values.

Table 1

The price–dividend ratio and surplus consumption for the economy in Example 2

This table summarizes qualitative properties of the continuous-time economy in Example 2. In this economy, a representative agent has habit formation preferences, with instantaneous utility $u(c, x) = (c - x)^\eta$, where η is the local curvature of the instantaneous utility, c is consumption, x is the habit stock, and $G_t \equiv y_t^{-\gamma}$ is solution to

$$dG_t = k(\bar{G} - G_t)dt - \alpha(G_t - l)\sigma_0 dW_{1t},$$

where W_{1t} is a Brownian motion, $y_t \equiv \frac{c_t - x_t}{c_t}$ is the surplus consumption ratio, σ_0 is consumption growth volatility, and γ, k, \bar{G} , and α are additional preference parameters related to the habit formation process. The properties in this table hold for $\gamma = 1$. They are obtained by applying the test conditions in Proposition 1a. The first column displays parameter restrictions. The second column lists qualitative features of the price–dividend ratio corresponding to the restrictions in the first column. The third column reports when the price-induced component of both return volatility and the price–dividend ratio volatility is asymmetric.

Parameter restriction	Price–dividend ratio	Price-induced asymmetric volatility
$\eta \in (0, 1)$ and $k\bar{G} > \alpha l \sigma_0^2 (\alpha + \eta - 1)$	Increasing and concave in y	Yes
$\eta = 1$	Increasing and linear in y	No
$\eta > 1$ and $k\bar{G} > \alpha l \sigma_0^2 \max\{\alpha + \eta - 1, (1 + \alpha)(\eta - 1)\}$	Increasing and convex in y	No

Example 2 generalizes two models in the literature: one, developed by [Menzly, Santos, and Veronesi \(MSV, 2004\)](#), in which the authors set $\gamma = \eta = 1$ (see also [Buraschi and Jiltsov, 2006](#)); the other, proposed by [Santos and Veronesi \(2006\)](#), in which $\gamma = \eta$. These parameter restrictions lead to closed-form solutions for the price–dividend ratio. For example, in the MSV economy, the price–dividend ratio for the aggregate consumption claim is linear in y . A natural question is: What happens in this economy if the preference parameters are such that $\gamma = 1$ (as in MSV), but the local utility curvature η is different from one? The answer can be obtained by applying the test conditions in Proposition 1a. It is summarized in [Table 1](#).

The economic intuition behind the restrictions in [Table 1](#) stems from the asymmetric behavior of the risk-adjusted discount rates $\text{Disc}(y_t) = R(y_t) + \sigma_0 \lambda_1(y_t)$. In this economy, the interest rate $R(y)$ and the unit risk premia are such that $\lambda_2(y) = 0$, and

$$\lambda_1(y) = \eta \sigma_0 [1 + (1 - ly)\alpha]; \quad \text{and} \tag{23}$$

$$R(y) = \rho + \eta g_0 - \frac{1}{2} \sigma_0^2 \eta (\eta + 1) + \eta k (1 - \bar{G}y) - \eta^2 \sigma_0^2 \alpha (1 - ly) - \frac{1}{2} \eta (1 - ly)^2 \alpha^2 \sigma_0^2 (\eta - 1). \tag{24}$$

While the risk premium $\lambda_1(y)$ is always decreasing and linear in the surplus consumption y_t , the asymmetric behavior of the discount rates $\text{Disc}(y)$ is affected by the local utility curvature η .

If $\eta = 1$, the discount rates $\text{Disc}(y)$ are decreasing and linear in y , i.e., they react symmetrically to changes in y_t . As a result, the price–dividend ratio is also increasing and linear in y_t . This property implies that the price-sensitivity $S_p(y_t) = p'(y_t)$ is constant and that the price-elasticity $E_p(y_t) = \frac{p'(y_t)}{p(y_t)}$ is countercyclical, with the countercyclical effect arising through the denominator. In other words, the price-sensitivity $S_p(y_t)$ is not countercyclical and so the price-induced component of the price–dividend ratio volatility is symmetric.

When is the price-sensitivity countercyclical? Proposition 1 identifies a precise condition: Either the discount rates or the risk-adjusted expectation about the future state of the economy, or both, must behave asymmetrically over some range of y_t . In this economy, the discount rates $\text{Disc}(y)$ have the desired property if $\eta \in (0, 1)$. This result follows because of the last term in the expression for the interest rate R in Eq. (24). This term reflects precautionary motives that become less important as the utility curvature parameter η decreases. Precisely, if η is less than one, the interest rate is decreasing and convex in the surplus ratio, i.e., it fluctuates more in bad times than in good. According to the explanations given in the Introduction (see Fig. 1), this asymmetry translates to countercyclical variation in the price-sensitivity, under the additional condition in Table 1 reflecting Condition a.2 in Proposition 1a. This is the prediction in the first row of Table 1.

The last row in Table 1 can be interpreted similarly: If the utility curvature parameter η is greater than one, the interest rate is decreasing and concave in y . This precautionary-induced asymmetry now makes the interest rate fluctuate more in good times than in bad, thereby leading to a procyclical variation in the price-sensitivity. Therefore, this model might or might not generate countercyclical variation in the price-sensitivity. According to Proposition 1a, this property crucially depends on the magnitude of the utility curvature η .

3.2. Restricted stock market participation

Countercyclical stock volatility could also arise in economies without habit formation. Consider the Basak and Cuoco (1998) model of restricted stock market participation. In this model, there are two agents. The first agent invests in the stock market and has an instantaneous utility function with constant relative risk-aversion equal to η . The second agent is prevented from investing in the stock market and has logarithmic preferences over consumption. The two agents have the same subjective discount rate ρ .

The marginal rate of substitution of the stock market participant is tied down to the pricing-kernel in Eq. (4) by the following relation:

$$\left(\frac{c_{p,t}}{c_{p,0}}\right)^{-\eta} = e^{\rho t} \zeta_t, \tag{25}$$

where $c_{p,t}$ is his optimal consumption. The marginal utility of the agent not participating in the stock market is $\left(\frac{c_{n,t}}{c_{n,0}}\right)^{-1} = e^{\rho t} e^{-\int_0^t R_s ds}$, where $c_{n,t}$ is his optimal consumption. By expanding the left- and right-hand sides of Eq. (25), by identifying terms, and by using the market clearing condition $c_{p,t} = D_t - c_{n,t}$, one finds that $\lambda_2(y) = 0$ and

$$\lambda_1(y_t) = \eta \cdot \text{Vol}\left(\frac{dc_{p,t}}{c_{p,t}}\right) = \eta \cdot \sigma_0 \frac{1}{y_t}, \tag{26}$$

where $y_t \equiv \frac{c_{p,t}}{D_t}$ is the market participant's consumption share and $\text{Vol}\left(\frac{dc_{p,t}}{c_{p,t}}\right) = \sigma_0 \frac{1}{y_t}$ is the instantaneous standard deviation of the market participant's consumption growth in equilibrium.

Intuitively, in this economy the stock market participant (i.e., the marginal investor) is bearing the entire macroeconomic risk. The risk premium he requires to invest in the stock market is large when his consumption share y_t is small. Moreover, the risk-adjusted discount rates $\text{Disc}(y_t) = R(y_t) + \sigma_0 \lambda_1(y_t)$ could display a similar property. By Basak and

Cuoco (1998, Corollary 1, p. 323),

$$R(y_t) = \rho + \frac{\eta g_0}{\eta - (\eta - 1)y_t} - \frac{1}{2} \eta(\eta + 1) \sigma_0^2 \frac{1}{\eta y_t - (\eta - 1)y_t^2}. \quad (27)$$

A simple computation reveals that, if $\eta > 1$, the discount rates $\text{Disc}(y)$ are convex in y and $\lim_{y \rightarrow 0} \text{Disc}(y) = \infty$, as in the Campbell and Cochrane habit formation economy of Example 1. In this economy with restricted stock market participation, the investor's evaluation of future cash flows is quite asymmetric. According to Proposition 1b, this asymmetry makes the price–dividend ratio fluctuate more on the downside, thereby producing asymmetric return volatility. Thus, both the Campbell and Cochrane (1999) economy and the Basak and Cuoco (1998) economy have an interesting property that helps illustrate the point of this paper: Countercyclical return volatility is induced by a pronounced asymmetry in the investors' attitude to discounting future cash flows.

3.3. Quantitative implications

I perform a calibration experiment to illustrate the quantitative implications of the theory. Table 2 displays the empirical benchmark for the calibration, based on a sample of 660 monthly observations from January 1948 through December 2002. The table reports the average and standard deviation for the price–dividend ratio on the Standard & Poor's (S&P) Composite index (P/D henceforth), the monthly changes $P/D_{t+1} - P/D_t$, the percentage changes of the P/D, defined as $\log(\frac{P/D_{t+1}}{P/D_t})$, the continuously compounded (real) returns, the riskless interest rate (the real one-month Treasury bill rate), and the excess return volatility.⁷ I compute the excess return volatility as of month t through the Officer (1973) moving standard deviation estimator, $\hat{\sigma}_t = \frac{1}{12} \sum_{i=1}^{12} |\text{Exc}_{t+1-i}|$, where Exc_t are the excess returns at month t . Then, I convert this volatility measure into $\widehat{\text{Vol}}_t \equiv \sqrt{\frac{\pi}{2}} \sqrt{12} \hat{\sigma}_t$. The $\sqrt{12}$ factor is used to annualize $\hat{\sigma}_t$; the $\sqrt{\frac{\pi}{2}}$ factor was suggested by Schwert (1989b, p. 1118) to correct a bias related to estimating the standard deviation through the absolute value of the excess returns.

Table 2 also reports descriptive statistics for the previous variables during National Bureau of Economic Research (NBER)-dated expansions and recessions. Return volatility is clearly countercyclical, as is the volatility of the P/D changes. For example, the excess return volatility is 0.14 on average (annualized). It increases by 22% during recessions and decreases by 4% during expansions. The P/D is procyclical but moves asymmetrically over the business cycle. According to the statistics in Table 2, the P/D is 32 on average, increases by 4% during expansions, and decreases by 18% during recessions. The volatility of the P/D changes is also asymmetric. It is 4.48 on average, increases by 26% during recessions, and decreases by 7% during expansions. Finally, the absolute value of the average of both the P/D changes and the P/D percentage changes is nearly twice as severe during recessions than expansions.

⁷The data on the P/D ratio are obtained from Robert Shiller's website. The remaining data are from the Center for Research in Security Prices database, with the exception of the seasonally adjusted Industrial Production, obtained from the FRED[®] database. Real returns and interest rates are obtained by deflating their nominal counterparts with the consumer price index.

Table 2

Business cycle properties of the price–dividend (P/D) ratio and returns

P/D is the Standard & Poor's (S&P) Composite price–dividend ratio. Real returns \tilde{R}_t are the log-returns on the S&P deflated by the consumer price index. Twelve month returns as of month t are defined as $\sum_{i=1}^{12} \tilde{R}_{t-i}$. Excess returns are returns in excess of the real (one month) risk-free rate and are computed similarly. Volatility is the excess return volatility, computed as $\widehat{\text{Vol}}_t \equiv \sqrt{\frac{\pi}{2} \sum_{i=1}^{12} \frac{|\text{Exc}_{t+i-i}|}{\sqrt{12}}}$, where Exc_t is the return in excess of the one-month bill return as of month t . Data are sampled monthly and cover the period from January 1948 through December 2002. All figures are annualized percent, with the exception of the P/D ratio levels and the changes $P/D_{t+1} - P/D_t$, which are only annualized. NBER: National Bureau of Economic Research.

Variables	Total		NBER expansions		NBER recessions	
	Average	Standard deviation	Average	Standard deviation	Average	Standard deviation
P/D	31.99	15.88	33.21	15.79	26.20	14.89
$P/D_{t+1} - P/D_t$	0.66	4.48	1.33	4.17	-2.54	5.64
$\log \frac{P/D_{t+1}}{P/D_t}$	2.01	12.13	3.95	10.81	-7.28	16.79
Real returns	8.22	14.94	9.70	13.82	1.17	19.28
Twelve month returns	8.59	15.86	12.41	13.04	-9.45	15.49
Real risk-free rate	1.02	2.48	1.03	2.43	0.97	2.69
Excess return volatility	14.18	4.86	13.50	4.49	17.38	5.18

The main point of this paper is that these properties can be explained by the asymmetric behavior of the risk premia over the business cycle. To provide empirical evidence in support of this theoretical finding, I use the Fama and French (1989) measurement procedure for estimating expected returns. I regress S&P returns (deflated by the consumer price index) on to the default-premium (Baa yield minus ten-year government bond yield), the term-premium (ten-year government bond yield minus three-month Treasury bill yield), and the return volatility $\widehat{\text{Vol}}_t$. The estimate of the expected returns at time t , $\hat{\mathcal{E}}_t$ say, is the fitted value at time t of this regression. Finally, I define one-year moving averages of the industrial production growth as $\text{IP}_t \equiv \frac{1}{12} \sum_{i=1}^{12} \text{Ind}_{t+1-i}$, where Ind_t is the real, seasonally adjusted industrial production growth as of month t .

The left-hand side of Fig. 3 plots the estimated expected returns $\hat{\mathcal{E}}_t$ against the industrial production growth IP_t . The right-hand side displays the fitted values of the least absolute deviations regression,

$$\hat{\mathcal{E}}_t = 8.56 - 4.05 \cdot \text{IP}_t + 1.18 \cdot \text{IP}_t^2 + w_t, \quad (28)$$

(0.15) (0.30) (0.31)

where w_t is a residual term and robust standard errors are in parenthesis.⁸ The evidence from Fig. 3 is striking. In good times, expected returns (a reasonable proxy for risk premia) do not vary much. In bad times, however, their fluctuations are more pronounced.

3.3.1. The benchmark economy

I now illustrate how the theory of this paper helps explain these large swings in the expected returns and return volatility. I use the habit formation economy in Example 2 to

⁸I run a least absolute deviations regression because this methodology is known to be more robust to the presence of outliers than ordinary least squares (see, e.g., Bloomfield and Steiger, 1983).

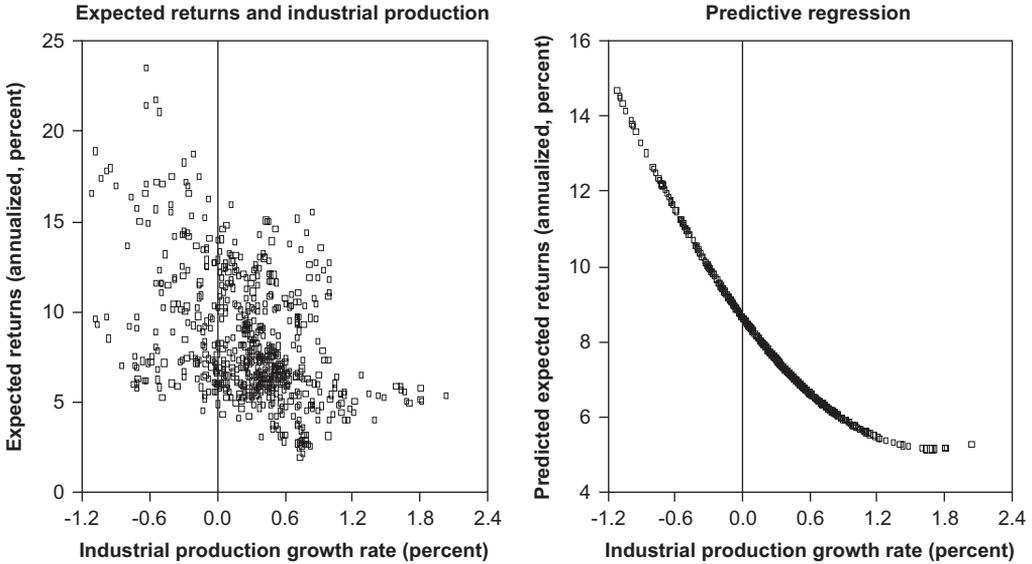


Fig. 3. Expected returns and business cycle conditions. The left-hand side of this picture plots estimates of the expected returns (annualized, percent) (\hat{e}_t , say) against one-year moving averages of the industrial production growth (IP_t). The expected returns are estimated through the predictive regression of Standard & Poor's returns on to default-premium, term-premium, and return volatility defined as $\widehat{Vol}_t \equiv \sqrt{\frac{\pi}{2} \sum_{i=1}^{12} \frac{|Exc_{t+1-i}|}{\sqrt{12}}}$, where Exc_t is the return in excess of the one-month bill return as of month t . The one-year moving average of the industrial production growth is computed as $IP_t \equiv \frac{1}{12} \sum_{i=1}^{12} Ind_{t+1-i}$, where Ind_t is the real, seasonally adjusted industrial production growth as of month t . The right-hand side of this picture depicts the prediction of the static least absolute deviations regression: $\hat{e}_t = 8.56 - 4.05 \cdot IP_t + 1.18 \cdot IP_t^2 + w_t$, where w_t is a residual term, and standard errors are in parenthesis. Data are sampled monthly and span the period from January 1948 to December 2002.

implement a calibration experiment. I specialize this economy to the case analyzed by Santos and Veronesi (2006), in which $\gamma = \eta$. In this economy, the interest rate $R(y)$ and the prices of risk are such that $\lambda_2(y) = 0$, and

$$\begin{aligned}
 R(y) &= \rho + \eta g_0 - \frac{1}{2} \sigma_0^2 \eta (\eta + 1) + k(1 - \bar{G}y^\eta) - \eta \alpha \sigma_0^2 (1 - ly^\eta), \quad \text{and} \\
 \lambda_1(y) &= \sigma_0 [\eta + \alpha(1 - ly^\eta)].
 \end{aligned}
 \tag{29}$$

If $\eta < 1$, the risk-adjusted discount rates $Disc(y) = R(y) + \sigma_0 \lambda_1(y)$ increase more in bad times than they decrease in good times, i.e., they are decreasing and convex in the surplus ratio y .⁹ Moreover, the sensitivity of the discount rates to changes in y_t can get arbitrarily large in bad times. Formally, if $\eta < 1$, then $\lim_{y \rightarrow 0} \frac{d}{dy} Disc(y) = \infty$, a property labeled “large asymmetry in discounting” in Proposition 1b.

According to Proposition 1, the previous properties translate to countercyclical movements in the price-elasticity and the price-sensitivity. This is confirmed analytically.

⁹Intuitively, λ_1 is always decreasing and convex. Moreover, low values of the utility curvature η make R fluctuate more in bad times than in good, because of (weak) intertemporal substitution effects (the fourth term in the expression for R). At the same time, low values of η mitigate the precautionary effects (the last term in the expression for R).

The solution for the price–dividend ratio is

$$p(y) = b_1 + b_2 y^\eta, \quad (30)$$

where b_1 and b_2 are two positive constants. That is, provided $\eta < 1$, the price–dividend ratio is concave in y and so the price–elasticity and the price–sensitivity are both countercyclical.

3.3.2. Calibration

I simulate the model discussed in Section 3.3.1 with one hundred samples of 50 years. I choose the model’s parameters so as to match its key unconditional population moments to the empirical counterparts in Table 2. These moments are the mean of the S&P Composite price–dividend ratio, the standard deviation of the price–dividend ratio changes and log–changes, and, finally, the mean and standard deviation of the continuously compounded returns and the risk–free rate. Table 3 reports the parameter values along with the calibration results. To generate countercyclical variation in the price–sensitivity, the utility curvature parameter η must be less than one, as discussed in Section 3.3.1. I use $\eta = \frac{1}{2}$. With the exception of this parameter, all parameter values are comparable with those in Santos and Veronesi (2006). The results in Table 3 show that the model generates figures for the average risky returns, the return volatility, and the mean and standard deviation of the risk–free rate, which are all very close to their empirical counterparts. Moreover, it successfully explains the level of the price–dividend ratio, along with the volatility of its changes.

Next, I look at the conditional moments implications of the model. I proceed as follows. First, I compute the certainty equivalent for the price–dividend ratio, defined as $\hat{y} : p(\hat{y}) = E[p(y_t)]$, where $E[\cdot]$ denotes the unconditional expectation operator. I define the “average states” of the economy as the states for which the surplus consumption ratio $y_t \in [y_-, y_+]$, where $y_{\mp} = \hat{y} \mp \frac{1}{2} \text{Std}(y)$ and $\text{Std}(y)$ is the unconditional standard deviation of y_t . I find that $\hat{y} = 1.97\%$ and $\text{Std}(y) = 0.59\%$. Next, I define the “good states” as those in which $y_t \in [y_+, y_+ + \Delta]$, where $\Delta = 0.50\%$. Finally, I define three levels of bad states. The less severe bad states occur when $y_t \in [y_- - \Delta, y_-]$ (bad states B1); the intermediate bad states occur when $y_t \in [y_- - 2\Delta, y_- - \Delta]$ (bad states B2); the most severe bad states occur when $y_t \in [y_- - 3\Delta, y_- - 2\Delta]$ (bad states B3).

Table 4 displays the key conditional population moments of the model. The oscillations of the price–dividend ratio from good states to bad are asymmetric and mimic the swings in the data. The model predicts that, as the price–dividend ratio moves away from the average states, it increases by 8% in the good states and decreases by 13% in the bad states B1. In particular, the price–sensitivity $S_p(y_t)$ increases by 18% in the bad states B1 and decreases by 8% in the good states. This asymmetry becomes more pronounced in the more severe states B2 and B3. Eventually, the price–sensitivity blows up as the surplus consumption ratio gets small. Intuitively, in this economy the representative agent modifies his discount rates dramatically as bad times deteriorate (the large asymmetry in discounting property discussed in Section 3.3.1), thereby inducing the price–sensitivity to change dramatically as well [Eq. (62) in the appendix formalizes this intuition].

The asymmetric behavior of the price–elasticity $E_p(y_t)$ is even more pronounced. This property accounts for the large movements in the expected returns shown in Table 4. This is because a large price–elasticity translates to a high price–dividend beta and so to large expected returns in Eq. (7). Finally, the model generates countercyclical return volatility, along with countercyclical variation in the volatility of the price–dividend ratio changes. The results in Table 4 reveal that the asymmetric movements in the price–sensitivity are

Table 3

Parameter values and basic calibration results for the model in Example 2

This table reports the basic calibration results for the infinite horizon, continuous-time economy in Example 2. Panel A has annualized parameter values used to match the basic moments of data reported in Panel B. The parameters g_0 and σ_0 are the instantaneous average and standard deviation of consumption growth, respectively. The remaining parameters affect the preferences of a representative agent with habit formation: ρ is the subjective discount rate, and η is the local curvature of the instantaneous utility $u(c, x) = (c - x)^\eta$, where c is consumption, x is the habit stock, and $G_t \equiv y_t^{-\gamma}$ is solution to

$$dG_t = k(\bar{G} - G_t)dt - \alpha(G_t - l)\sigma_0 dW_{1t},$$

where W_{1t} is a Brownian motion and $y_t \equiv \frac{c_t - x_t}{c_t}$ is the surplus consumption ratio. In this calibration experiment, γ is set equal to η . Panel B has calibration results for the average Standard & Poor's (S&P) Composite price–dividend ratio, P/D ; the standard deviation of the P/D changes, $\text{Std}(P/D_{t+1} - P/D_t)$; the standard deviation of the log- P/D changes, $\text{Std}(\log \frac{P/D_{t+1}}{P/D_t})$; the average real log-returns on the S&P (deflated by the consumer price index), $E(\tilde{R})$; the standard deviation of returns, $\text{Std}(\tilde{R})$; the average real (one-month) risk-free rate $E(r^f)$; and the standard deviation of the risk-free rate, $\text{Std}(r^f)$. All figures in Panel B are annualized percent, with the exception of the P/D ratio levels, and the changes $P/D_{t+1} - P/D_t$, which are only annualized.

Panel A. Consumption and preference parameters

g_0	σ_0	η	ρ	\bar{G}	k	l	α
0.020	0.015	$\frac{1}{2}$	0.045	8.36	0.16	6.01	73.85

Panel B. Moments of historical data and moments implied by the model

Sample data	$E(P/D)$	$\text{Std}(P/D_{t+1} - P/D_t)$	$\text{Std}(\log \frac{P/D_{t+1}}{P/D_t})$	$E(\tilde{R})$	$\text{Std}(\tilde{R})$	$E(r^f)$	$\text{Std}(r^f)$
Historical data	31.99	4.48	12.13	8.22	14.94	1.02	2.48
Model	32.27	3.75	13.45	7.27	14.95	2.08	3.51

Table 4

Conditional moments implied by the calibrated model in Example 2

Consumption and preference parameters are fixed at the values in Table 3, Panel A. The P/D sensitivity is the sensitivity of the price–dividend (P/D) ratio to changes in the surplus consumption ratio y and is computed as $p'(y)$, where p is the P/D ratio. The P/D elasticity is the elasticity of the P/D ratio to changes in y , computed as $\frac{p'(y)}{p(y)}$. The P/D volatility is the volatility of the instantaneous changes in the P/D ratio, computed as $p'(y)v(y)$, where $v(y)$ is the instantaneous volatility of y . The log-P/D volatility is the volatility of the instantaneous changes in the log-P/D, computed as $\frac{p'(y)}{p(y)}v(y)$. Surplus volatility is the instantaneous volatility of the surplus ratio, $v(y)$. The risk-adjusted rates are the discount rates adjusted for cash flow risk. The table reports expectations conditional on the surplus ratio y belonging to pre-specified states. For each variable, its expectation in the average states is the expectation conditional on y belonging to the interval $[y_-, y_+]$, where $y_{\mp} = \hat{y} \mp \frac{1}{2}\text{Std}(y)$; $\text{Std}(y)$ is the standard deviation of y (with $\text{Std}(y) = 0.59 \cdot 10^{-2}$); and \hat{y} is the certainty equivalent for the P/D ratio, defined as $\hat{y} : p(\hat{y}) = E[p(y_t)]$ (with $\hat{y} = 1.97 \cdot 10^{-2}$). For each variable, the expectation in the good states is the expectation conditional on y belonging to $[y_+, y_+ + \Delta]$, where $\Delta = 0.50 \cdot 10^{-2}$. Finally, the expectation in the bad states B1, B2, and B3 is the expectation conditional on y belonging to $[y_- - n\Delta, y_- - (n-1)\Delta]$, for $n = 1$ (the bad states B1), $n = 2$ (the bad states B2), and $n = 3$ (the bad states B3), respectively. The figures for the log-P/D volatility, return volatility, risk-adjusted rates, and expected returns are annualized percent.

Variables	Good states	Average states	Bad states		
			B1	B2	B3
P/D	35.52	32.62	28.29	23.81	17.58
P/D sensitivity	618.18	675.56	796.52	976.03	1450.21
P/D elasticity	17.33	20.46	27.72	40.15	81.52
P/D volatility	1.72	4.02	6.89	8.48	8.57
log-P/D volatility	4.86	12.27	24.02	34.96	47.06
Surplus volatility	2.81×10^{-3}	5.97×10^{-3}	8.66×10^{-3}	8.68×10^{-3}	5.87×10^{-3}
Return volatility	6.36	13.79	25.52	36.52	48.80
Risk-adjusted rates	0.53	2.61	5.68	8.87	13.31
Expected returns	1.36	6.81	16.76	31.73	49.61

quantitatively responsible for the bulk of variation in the volatility of the price–dividend ratio changes. To summarize, in this economy the discount rates react asymmetrically to changes in economic conditions. Furthermore, this asymmetry becomes more pronounced as bad times worsen. Importantly, the calibration exercise reveals that the theoretical issues this paper associates with such asymmetries can have substantial quantitative implications on the dynamics of expected returns and volatility.

4. Economies with fluctuating uncertainty

How do the restrictions in this paper work in a world with fluctuating economic uncertainty? I now analyze economies in which consumption growth is surrounded by varying uncertainty.

4.1. The dynamics of fundamentals and asset prices

I consider an economy in which expected consumption growth and consumption volatility are both time-varying,

$$\frac{dD_t}{D_t} = x_t dt + \sigma_D(s_t) dW_{1t}, \quad (31)$$

where D_t is the consumption endowment, x_t is the expected consumption growth, and s_t is a state variable affecting consumption volatility $\sigma_D(s_t)$, for some positive function σ_D increasing in s_t . Both expected consumption growth and consumption volatility are predictable:

$$\begin{aligned} dx_t &= \gamma(\bar{x} - x_t) dt + \sigma_x dW_{2t}, \\ ds_t &= \kappa(\bar{s} - s_t) dt + \sigma_s(s_t, q_t) dW_{3t}, \\ dq_t &= \beta(\bar{q} - q_t) dt + \sigma_q(q_t) dW_{4t}, \end{aligned} \tag{32}$$

for some additional Brownian motions W_2, W_3 , and W_4 ; some positive constants $\gamma, \bar{x}, \sigma_x, \kappa, \bar{s}, \beta$, and \bar{q} ; and some functions σ_s and σ_q , which are assumed to be increasing in their arguments.

These assumptions about consumption volatility are the continuous-time counterpart to Tauchen (2005). Tauchen’s assumptions about volatility extend upon those in Bansal and Yaron (2004) because the function $\sigma_s(s_t, q_t)$ could depend on s_t and some additional volatility of volatility state variable q_t . In turn, the previous model is a slight generalization of Tauchen’s as it includes variation in the expected consumption growth x_t . However, the expected growth x_t does not play any role in the analysis below.

As in Section 2, I assume that the interest rate and risk premia are independent of the level of aggregate consumption D_t . Therefore, the pricing kernel ξ_t is the solution to

$$\frac{d\xi_t}{\xi_t} = -R(\mathbf{y}_t) dt - \lambda(\mathbf{y}_t) d\mathbf{W}_t, \tag{33}$$

where $\mathbf{y}_t = [x_t, s_t, q_t]^\top$, $\mathbf{W} = [W_1, \dots, W_4]^\top$, and, finally, $\lambda = [\lambda_D, \lambda_x, \lambda_s, \lambda_q]$ is the vector of unit prices of risk. These assumptions imply that the price–dividend ratio $\frac{P_t}{D_t} = p(\mathbf{y}_t)$, for some function p . I assume that R and λ are twice continuously differentiable and that the price–dividend ratio p satisfies the same regularity conditions as in Section 2. To keep the presentation simple, I also assume that every unit price of risk λ_i depends only on the variables that affect the volatility of the state variable i ($i = D, x, s, q$), i.e., $\lambda_D \equiv \lambda_D(s)$, $\lambda_x \equiv \lambda_x(x)$, $\lambda_s \equiv \lambda_s(s, q)$ and $\lambda_q \equiv \lambda_q(q)$.¹⁰

Which conditions should the pricing kernel satisfy to make returns inversely related to the volatility of fundamentals s_t and the volatility of volatility q_t ? By the definition of asset returns,

$$\text{Returns}_t = \frac{dP_t + D_t dt}{P_t} = \mathcal{E}(\mathbf{y}_t) dt + \text{Vol}(\mathbf{y}_t) d\mathbf{W}_t, \tag{34}$$

where $\text{Vol} = [\text{Vol}_1, \dots, \text{Vol}_4]$, $\text{Vol}_1 = \sigma_D$, $\text{Vol}_i = \frac{\partial_i p}{p} \sigma_i$ ($i = x, s, q$), subscripts on the price–dividend ratio p denote partial derivatives (for example, $p_s \equiv \frac{\partial p(x, s, q)}{\partial s}$), and, finally, the expected returns in Eq. (34) are obtained similarly as in Eq. (7) in Section 2. They are

$$\mathcal{E}(\mathbf{y}_t) = \underbrace{R(\mathbf{y}_t) + \sigma_D(s_t)\lambda_D(s_t)}_{\equiv \text{Disc}(\mathbf{y}_t)} + \sum_{i=x,s,q} \text{Vol}_i(\mathbf{y}_t)\lambda_i(\mathbf{y}_t). \tag{35}$$

By Eqs. (32) and (34),

$$E_t(\text{Returns}_t \cdot d\ell_t) = \frac{p_i(\mathbf{y}_t)}{p(\mathbf{y}_t)} \sigma_i(\mathbf{y}_t)^2 dt, \quad \ell_t = s_t \text{ or } q_t. \tag{36}$$

¹⁰The interest rate and the risk premia in Eq. (33) do not depend on some additional index tracking the state of the economy [such as that in Eq. (3)]. Adding this would lead to the same analysis as in Section 2, without affecting the results in this section.

Thus, ex post returns co-move negatively with the volatility factors s_t and q_t if the price–dividend ratio p decreases with s_t and q_t , i.e., $p_s < 0$ and $p_q < 0$. Proposition 2 develops conditions under which such inverse relations do occur.

Proposition 2. *Let the endowment process be as in Eq. (31) and suppose that the risk-adjusted discount rates in Eq. (35) are increasing in the volatility of fundamentals s , i.e., $\frac{\partial}{\partial s} \text{Disc}(\mathbf{y}) > 0$. The following statements hold true under the assumptions in this section*

- (a) *The price–dividend ratio reacts negatively to changes in the volatility of fundamentals s , i.e., $p_s(\mathbf{y}) < 0$. Hence, the asset returns in Eq. (34) and s_t are negatively correlated.*
- (b) *Assume that the volatility risk premia are negative and increasing in q , i.e., $\lambda_s(s, q) < 0$ and $-\frac{\partial[\lambda_s(s, q)\sigma_s(s, q)]}{\partial q} > 0$, and that one of the two following conditions holds true:*
 - (b.1) *The discount rates are increasing in the volatility of volatility q , i.e. $\frac{\partial}{\partial q} \text{Disc}(\mathbf{y}) > 0$. In addition, (i) the volatility of fundamentals $s \in (\underline{s}, \bar{s})$, for two positive constants \underline{s} and \bar{s} , and (ii) the price–dividend ratio reacts asymmetrically to changes in s , i.e., it is concave in s .*
 - (b.2) $-\frac{\partial \text{Disc}}{\partial q} p - \frac{\partial(\lambda_s \sigma_s)}{\partial q} p_s + \frac{1}{2} \frac{\partial \sigma_s^2}{\partial q} p_{ss} < 0$ (Large negative volatility risk premia).

Then, the price–dividend ratio is decreasing in the volatility of volatility q , i.e., $p_q(\mathbf{y}) < 0$. Hence, the asset returns in Eq. (34) and q_t are negatively correlated.

Conversely, suppose that the price–dividend ratio is decreasing in the volatility of fundamentals s (respectively, the volatility of volatility q). Then, the discount rates $\text{Disc}(\mathbf{y})$ must be increasing in s (respectively, Condition b.2 must hold) on some sets of \mathbf{y} having strictly positive probability.

As in Sections 2 and 3, the discount rates $\text{Disc}(\mathbf{y}_t)$ in Eq. (35) play a critical role. Consider Part (a) of Proposition 2. It relies on the assumption that, after a positive shock to the volatility of fundamentals s_t , investors raise the discount rates they use to evaluate future dividends. Under this condition, an increase in the volatility of fundamentals lowers the price–dividend ratio and induces a negative relation between asset returns and the volatility of fundamentals.

Part (b) of Proposition 2 relates asset returns to higher order properties of the economic fundamentals. When is the price–dividend ratio inversely related to the volatility of volatility q_t ? First, volatility risk should be negatively priced, i.e., $\lambda_s < 0$. Intuitively, if prices are negatively affected by volatility, a negative volatility risk premium is required to make Arrow–Debreu state prices high in the poor states of the world (i.e., when volatility is high). Second, the volatility risk premia should increase with the volatility of volatility q , i.e., $\frac{\partial}{\partial q}(-\lambda_s \sigma_s) > 0$. In other words, following a positive shock to q_t , the compensation for volatility fluctuations should increase, thus lowering the asset price and returns. These two basic conditions are not sufficient to make the price–dividend ratio decreasing in q_t . Proposition 2 identifies two additional sets of conditions.

1. Condition b.1 is a sufficient condition. It requires that the discount rates $\text{Disc}(\mathbf{y}_t)$ be increasing in q_t and that the price–dividend ratio be decreasing and concave in the volatility of fundamentals s_t . Intuitively, the concavity property means that the price–dividend ratio dampens low realizations of s_t and exaggerates higher realizations

of s_t . As a result, a positive spread in the uncertainty surrounding the volatility s_t (i.e., an increase in the volatility of the volatility q_t) lowers the price–dividend ratio, a conclusion consistent with second-order stochastic dominance. In turn, the proof of Proposition 2 reveals that the following conditions guarantee that the price–dividend ratio is concave in s :

$$\frac{\partial^2}{\partial s^2} \text{Disc}(\mathbf{y}) > 0 \quad (\text{Asymmetric discount rates}) \tag{37}$$

and

$$-\frac{\partial^2}{\partial s^2} [\lambda_s(s, q)\sigma_s(s, q)] > 2\frac{\partial}{\partial s} \text{Disc}(\mathbf{y}) \quad (\text{Asymmetric volatility risk premium}). \tag{38}$$

These conditions require that the discount rates $\text{Disc}(\mathbf{y}_t)$ and the volatility risk premia $-\lambda_s(s_t, q_t)\sigma_s(s_t, q_t)$ increase more in bad times (when the volatility s_t is high) than they decrease in good times (when the volatility s_t is low). Similarly as in Sections 2 and 3, this kind of asymmetry implies that the price–dividend ratio p fluctuates more in bad times than in good, i.e., it is decreasing and concave with respect to s .

2. Condition b.2 holds under Condition b.1, but the converse is clearly not true. Moreover, the last part of Proposition 2 states that the necessary condition for the price–dividend ratio to be decreasing in q is that Condition b.2 holds on some sets of \mathbf{y} having positive probability. In particular, Condition b.2 is satisfied if the volatility risk premia are sufficiently responsive to changes in q_t (i.e., if the sensitivity $-\frac{\partial(\lambda_s\sigma_s)}{\partial q}$ is sufficiently large). Section 4.2 provides examples of economies in which volatility risk premia behave in the manner prescribed by Condition b.2.

Finally, the feedback effect discussed in the Introduction arises if the return volatility components in Eq. (34) increase after a positive shock in s_t and in q_t . In all the economies I consider in Section 4.2, this property arises under the same conditions stated in Proposition 2.

4.2. An application to nonexpected utility

I use Proposition 2 to analyze the economies considered by [Bansal and Yaron \(2004\)](#) and [Tauchen \(2005\)](#). In these two papers, the primitives satisfy a discrete-time version of Eqs. (31) and (32), and a representative investor is endowed with the [Epstein and Zin \(1989\)](#) and [Weil \(1989\)](#) nonexpected, but recursive utility. Consumption growth satisfies

$$\log D_{t+\Delta t} - \log D_t = \left(x_t - \frac{1}{2}\sigma_D(s_t)^2 \right) \Delta t + \sigma_D(s_t)\varepsilon_{1,t+\Delta t}\sqrt{\Delta t}, \tag{39}$$

where $\Delta t > 0$, $\varepsilon_{1,t}$ is independent and identically distributed as a standard normal variable, and expected consumption growth x_t , consumption volatility s_t , and the volatility of volatility q_t are the discrete-time counterparts to Eqs. (32) [see Eqs. (69) in the appendix].

4.2.1. The risk-free rate and the risk premia

The asset price P_t satisfies the Euler equation $\xi_t P_t = E_t[\xi_{t+\Delta}(P_{t+\Delta} + D_t \cdot \Delta t)]$, where the pricing kernel ξ_t is defined recursively as $\xi_{t+\Delta} \equiv M_{t+\Delta} \xi_t$ (with $\xi_0 = 1$) and M_t is the stochastic discount factor. In the [Epstein and Zin](#) and [Weil](#) environment, the stochastic

discount factor satisfies

$$\log \xi_{t+\Delta t} - \log \xi_t = \log M_{t+\Delta t} = -\theta \rho \Delta t - \frac{\theta}{\psi} \log \left(\frac{D_{t+\Delta t}}{D_t} \right) + (\theta - 1) \log \left(\frac{P_{t+\Delta t} + D_t \Delta t}{P_t} \right), \tag{40}$$

where $\theta \equiv \frac{1-\eta}{1-\frac{1}{\psi}}$, ρ is the subjective discount rate, η is the relative risk-aversion for static gambles, and ψ is the intertemporal elasticity of substitution (IES, henceforth). In the standard expected utility framework, $\eta = \psi^{-1}$.

In the appendix, I show that in the continuous-time limit, the pricing kernel in Eq. (40) satisfies Eq. (33), where the interest rate is

$$R(\mathbf{y}_t) = \rho + \frac{1}{\psi} x_t - \frac{1}{2} \eta \left(1 + \frac{1}{\psi} \right) \sigma_D(s_t)^2 - \frac{1}{2} (1 - \theta) \sum_{i=x,s,q} \left(\sigma_i(\mathbf{y}_t) \frac{p_i(\mathbf{y}_t)}{p(\mathbf{y}_t)} \right)^2, \tag{41}$$

and the vector of unit-risk premia $\lambda = [\lambda_D, \lambda_x, \lambda_s, \lambda_q]$, with $\lambda_D = \eta \sigma_D$ and $\lambda_i = (1 - \theta) \sigma_i \frac{p_i}{p}$ ($i = x, s, q$). Therefore, in this economy, the discount rates in Eq. (35) are

$$\text{Disc}(\mathbf{y}_t) = \rho + \frac{1}{\psi} x_t + \frac{1}{2} \eta \left(1 - \frac{1}{\psi} \right) \sigma_D(s_t)^2 - \frac{1}{2} (1 - \theta) \sum_{i=x,s,q} \left(\sigma_i(\mathbf{y}_t) \frac{p_i(\mathbf{y}_t)}{p(\mathbf{y}_t)} \right)^2. \tag{42}$$

These expressions for the interest rate and the risk premia appear to be new to the literature. I now use them to check the test conditions in Proposition 2.

4.2.2. Ex post returns and return volatility

I relate ex post returns and return volatility to changes in the volatility of fundamentals s_t and in the volatility of volatility q_t .

When are prices inversely related to the volatility of fundamentals s_t ? According to Proposition 2a, this property arises if the discount rates $\text{Disc}(\mathbf{y}_t)$ are increasing in s_t . Consider, for example, the expected utility case in which $\eta = \frac{1}{\psi}$ and, hence, $\theta = 1$. Eq. (42) reveals that, if precautionary motives are not too strong (i.e., if the IES $\psi > 1$), the investor raises his discount rates after a positive shock to the volatility s_t , thereby inducing an inverse relation between s_t and ex post returns. Moreover, by continuity, the same inverse relation obtains for a fixed IES $\psi > 1$ and some values of η higher than $\frac{1}{\psi}$.¹¹

To further analyze the case $\theta < 1$, we need to understand how the last term in the right-hand side of Eq. (42) changes with s_t . Consider a log-linear expansion of the price–dividend ratio such that $\sigma_i \frac{p_i}{p} \approx \sigma_i A_i$, for three positive constants A_i , $i = s, q, x$. By substituting this approximation into Eq. (42),

$$\text{Disc}(\mathbf{y}_t) = \rho + \frac{1}{\psi} x_t + \frac{1}{2} \eta \left(1 - \frac{1}{\psi} \right) \sigma_D(s_t)^2 - \frac{1}{2} (1 - \theta) \sum_{i=x,s,q} A_i^2 \sigma_i(\mathbf{y}_t)^2. \tag{43}$$

The difference $1 - \theta = (1 - \frac{1}{\psi})^{-1} (\eta - \frac{1}{\psi})$ plays a crucial role in Eq. (43). Let the IES $\psi > 1$ (the case $\psi < 1$ can be analyzed in a similar way). If the risk-aversion parameter $\eta > 1$, the last term on the right-hand side of Eq. (43) decreases with the volatility s_t .

¹¹The result that $p_s < 0$ for $\theta = 1$ differs from those in Bansal and Yaron and in Tauchen. The difference arises because to make Eq. (39) consistent with its continuous time counterpart in Eq. (31), I corrected expected consumption growth for Jensen’s inequality effects through the additional term $\frac{1}{2} \sigma_D(s_t)^2$.

What is the economic intuition for this result? From the work of [Kreps and Porteus \(1978\)](#) and [Epstein and Zin \(1989\)](#), we know that if $\eta > \frac{1}{\psi}$, then the investor has a preference for early resolution of uncertainty. This preference induces him to accelerate his consumption plan in response to increased uncertainty. In general equilibrium, this fall in his (planned) expected consumption growth is possible with a fall in the interest rate in Eq. (41) and hence in the discount rates in Eqs. (42) and (43). Therefore, to make discount rates increasing in the volatility s (as required by Proposition 2), the term $\sum_{i=x,s,q} A_i^2 \sigma_i(\mathbf{y}_t)^2$ in Eq. (43) should be relatively insensitive to changes in s . Equivalently, the discount rates in Eq. (43) are increasing in s_t when the volatility risk premium $\lambda_s = (1 - \theta)\sigma_s \frac{p_s}{p} \approx A_s(1 - \theta)\sigma_s$ is not too responsive to changes in s .

These properties arise if the volatility of volatility function $\sigma_s(s, q)$ is independent of s , as in [Bansal and Yaron \(2004\)](#) and in the two-factor setting in [Tauchen \(2005, Section 3\)](#). They also arise in the one-factor model of [Tauchen \(2005, Section 2\)](#), in which $\sigma_D(s) \equiv \sqrt{s}$ and $\sigma_s(s, q) \equiv \phi_s \sqrt{s}$, provided the positive constant ϕ_s is not too large.¹² Under these conditions, variation in s also leads to the volatility feedback. This is because by Eq. (34), the return volatility component related to s is approximately $(\frac{p_s}{p} \sigma_s)^2 \approx A_s^2 \sigma_s^2$ and is increasing in s .

How do prices react to changes in the volatility of the volatility q_t ? By Eq. (43), the preference for early resolution of uncertainty implies that the discount rates are increasing in q_t . Therefore, Condition b.1 in Proposition 2b does not hold. Instead, the price–dividend ratio is decreasing in q_t if Condition b.2 holds. In turn, Condition b.2 is satisfied if the sensitivity of the volatility risk premia with respect to changes in q_t , $-\frac{\partial(\lambda_s \sigma_s)}{\partial q}$, is large enough to dwarf the previous uncertainty resolution effects.

Volatility risk premia can have this property in the Tauchen two-factor model, in which $\sigma_s(s, q) = \sqrt{q}$, $\sigma_q(q) = \phi_q \sqrt{q}$ and $\lambda_s(s, q) \approx A_s(1 - \theta)\sqrt{q}$. In this model, the sensitivity $\frac{\partial(\lambda_s \sigma_s)}{\partial q} \approx \frac{1}{2}(1 - \theta)A_s$ and should be large compared with the uncertainty parameter ϕ_q .¹³ Proposition 2b predicts that, in this case, ex post returns decrease after a positive shock to the volatility of the volatility q_t . Moreover, changes in q_t raise return volatility as the volatility component related to q is approximately $(\frac{p_q}{p} \sigma_q)^2 \approx A_q^2 \sigma_q^2$ and is increasing in q . Hence, Proposition 2b predicts that, in this economy, the feedback effect can be induced by variation in the volatility of volatility. By the previous discussion, this effect arises when the reaction of the volatility risk premium to changes in q_t is large enough to mitigate the effects of a preference for early resolution of uncertainty.

5. Conclusion

Why is stock market volatility higher in bad times than in good times? One possible explanation is that the economy is frequently hit by shocks with the same properties as those ultimately observed in the asset prices. Another possibility is that stock market

¹²It is easily seen that in the Tauchen’s one-factor model, $\frac{\partial}{\partial s} \text{Disc} > 0$ if and only if $\frac{1}{2}\eta(1 - \frac{1}{\psi}) - \frac{1}{2}(1 - \theta)A_s^2 \phi_s^2 > 0$, i.e., if and only if ϕ_s is not too large and $\lim_{\phi_s \rightarrow 0} A_s^2 \phi_s^2 = 0$.

¹³By replacing the expressions for σ_s , σ_q , and λ_s and the log-linear approximation $p_i \approx A_i p$ into Condition b.2, I find that this condition holds if $\frac{1}{2}(1 - \theta)A_q^2 \phi_q^2 - \frac{1}{2}(1 - \theta)A_s^2 + \frac{1}{2}A_s^2 < 0$, i.e., if the parameter ϕ_q is not too large and $\lim_{\phi_q \rightarrow 0} A_q^2 \phi_q^2 = 0$. Tauchen emphasizes the importance of a similar condition.

volatility is countercyclical as a result of rational asset evaluation. The explanations provided in this article rely upon some fundamental facts that underlie rational asset evaluation. I find that countercyclical return volatility is induced by large swings of risk premia that occur when the economy moves away from good states. The logic behind this explanation is intuitive. If asset prices are risk-adjusted, discounted expectations of future dividends, then these expectations are worse in bad times than in good times. If changes in these discounted expectations (and, hence, in the risk premia) are also more pronounced in bad times than in good times, then price volatility is countercyclical.

This channel of asymmetric volatility relies on a framework in which the uncertainty about the fundamentals of the economy is fixed and countercyclical stock market volatility is induced by asymmetric movements of the risk premia. But I also analyze economies in which the fundamentals are surrounded by fluctuating uncertainty, and study if and how the risk premia for this uncertainty lead return volatility to be higher in bad times than in good.

My results hold for a fairly rich class of dynamic economies. For this reason, they accomplish two tasks. First, they provide fresh directions into the search process for the determinants of asymmetric volatility. Second, they highlight new asymmetric volatility channels that any model should feature to be consistent with rational asset evaluation. Models that do not activate these channels are likely to fail on one important dimension of actual stock market fluctuations: the systematic occurrence of countercyclical movements in return volatility.

Appendix

Notation. For any function f of a single variable x , I let $f'(x) \equiv \frac{d}{dx}f(x)$ and $f''(x) \equiv \frac{d^2}{dx^2}f(x)$. Moreover, for any vector $a \equiv [a_1 \cdots a_N]$, I let $\|a\|^2 \equiv a_1^2 + \cdots + a_N^2$.

Derivation of Eq. (7). To pin down $\mathcal{E}(y)$ in Eq. (7), I develop the pricing equation $0 = \xi_t D_t dt + E_t[d(\xi_t P_t)]$, obtaining

$$\begin{aligned} E_t\left(\frac{dP_t + D_t dt}{P_t}\right) &= R(y_t) dt - cov_t\left(\frac{dP_t}{P_t}, \frac{d\xi_t}{\xi_t}\right) \\ &= R(y_t) dt - cov_t\left(\frac{dD_t}{D_t}, \frac{d\xi_t}{\xi_t}\right) - cov_t\left(\frac{dp(y_t)}{p(y_t)}, \frac{d\xi_t}{\xi_t}\right) \\ &= [R(y_t) + \beta_{CF}\lambda_1(y_t) + \beta_{P/D}(y_t) \cdot \lambda(y_t)] dt, \end{aligned} \tag{44}$$

where the second line follows by the definition of the price–dividend ratio $p(y_t) = \frac{P_t}{D_t}$, the third line follows by Eqs. (4) and (6), and β_{CF} and $\beta_{P/D}$ are as in the main text. \square

The proof of Proposition 1 in Section 2 relies on the following preliminary result.

Lemma 1. *Let $\{y_t\}_{t \geq 0}$ be the (strong) solution to*

$$dy_t = b(y_t) dt + a(y_t) d\hat{W}_t, \tag{45}$$

where \hat{W} is a multidimensional Q -Brownian motion, and b and a are some given functions (a is vector-valued). Let ρ and ψ be two positive twice continuously differentiable functions, and define,

$$c(y, T) \equiv \mathbb{E}\left[\exp\left(-\int_0^T \rho(y_t) dt\right) \psi(y_T) \Big| y_0 = y\right]. \tag{46}$$

The following statements are true

- (a) If $\psi' > 0$, c is increasing in y whenever $\rho' \leq 0$. If $\psi' = 0$, c is decreasing in y whenever $\rho' > 0$.
- (b) If $\psi'' \leq 0$ (respectively, $\psi'' \geq 0$) and c is increasing (respectively, decreasing) in y , c is concave (respectively, convex) in y whenever $b'' < 2\rho'$ (respectively, $b'' > 2\rho'$) and $\rho'' \geq 0$ (respectively, $\rho'' \leq 0$). Finally, if $b'' = 2\rho'$, c is concave (respectively, convex) in y whenever $\psi'' < 0$ (respectively, > 0) and $\rho'' \geq 0$ (respectively, ≤ 0).

Proof. Let $c(y, T - s) \equiv \mathbb{E}[\exp(-\int_s^T \rho(y_t) dt) \cdot \psi(y_T) \mid y_s = y]$. The function c is solution to

$$\begin{cases} 0 = -c_2(y, T - s) + \mathcal{L}c(y, T - s) - \rho(y)c(y, T - s), & \forall (y, s) \in \mathbb{R} \times [0, T), \\ c(y, 0) = \psi(y), & \forall y \in \mathbb{R}, \end{cases} \tag{47}$$

where $\mathcal{L}c(y, u) = \frac{1}{2}\|a(y)\|^2 c_{yy}(y, u) + b(y)c_y(y, u)$, $c_2(y, u) \equiv c_{uu}(y, u)$, and subscripts denote partial derivatives. By differentiating Eq. (47) twice with respect to y , I find that $c^{(1)}(y, t) \equiv c_y(y, t)$ and $c^{(2)}(y, t) \equiv c_{yy}(y, t)$ are solutions to the following partial differential equations:

$$\begin{aligned} 0 = & -c_2^{(1)}(y, T - s) + \frac{1}{2}\|a(y)\|^2 c_{yy}^{(1)}(y, T - s) + \left[b(y) + \frac{1}{2} \frac{d}{dy} \|a(y)\|^2 \right] c_y^{(1)}(y, T - s) \\ & - [\rho(y) - b'(y)]c^{(1)}(y, T - s) - \rho'(y)c(y, T - s), \end{aligned} \tag{48}$$

where $c^{(1)}(y, 0) = \psi'(y)$ and

$$\begin{aligned} 0 = & -c_2^{(2)}(y, T - s) + \frac{1}{2}\|a(y)\|^2 c_{yy}^{(2)}(y, T - s) + \left[b(y) + \frac{d}{dy} \|a(y)\|^2 \right] c_y^{(2)}(y, T - s) \\ & - \left[\rho(y) - 2b'(y) - \frac{1}{2} \frac{d^2}{dy^2} \|a(y)\|^2 \right] c^{(2)}(y, T - s) \\ & - [2\rho'(y) - b''(y)]c^{(1)}(y, T - s) - \rho''(y)c(y, T - s), \end{aligned} \tag{49}$$

where $c^{(2)}(y, 0) = \psi''(y)$.

By the Feynman and Kac theorem, the solution to Eq. (48) is

$$c^{(1)}(y, T - s) = \mathbb{E} \left[\int_s^T \kappa(s, \tau) (-\rho'(y_\tau)) c(y_\tau, T - \tau) d\tau \mid y_s = y \right] + \mathbb{E}[\kappa(s, T)\psi'(y_T) \mid y_s = y], \tag{50}$$

where $\kappa(s, \tau) = \exp\{-\int_s^\tau [\rho(y_u) - b'(y_u)] du\}$, and y is the solution to Eq. (45), but with drift equal to $b + \frac{1}{2} \frac{d}{dy} \|a\|^2$ [which is the drift multiplying $c_y^{(1)}$ in Eq. (48)]. Hence, $c^{(1)}(y, T - s) > 0 \forall (y, s) \in \mathbb{R} \times [0, T)$ whenever $\rho'(y) < 0$ and $\psi'(y) > 0 \forall y \in \mathbb{R}$. This completes the proof of Part (a) of Lemma 1. The proof of Part (b) is obtained similarly. \square

Proof of Proposition 1 (Part (a)). By Eq. (1), the price–dividend ratio satisfies

$$p(y) = \int_0^\infty B(y, t) dt, \tag{51}$$

where

$$\begin{aligned}
 B(y, t) &= \mathbb{E} \left[\beta_t \cdot \exp \left(- \int_0^t (R(y_u) + \sigma_0 \lambda(y_u) - g_0) du \right) \middle| y_0 = y \right] \\
 &\equiv \bar{\mathbb{E}} \left[\exp \left(- \int_0^t (R(y_u) + \sigma_0 \lambda(y_u) - g_0) du \right) \middle| y_0 = y \right], \tag{52}
 \end{aligned}$$

$\mathbb{E}[\cdot]$ is the expectation under the risk-neutral probability Q , $\bar{\mathbb{E}}[\cdot]$ is the expectation taken under a new measure \bar{Q} defined as $d\bar{Q}/dQ = \beta_t \equiv \exp(-\frac{1}{2}\sigma_0^2 t + \sigma_0 \hat{W}_{1t})$, and \hat{W}_1 is the Brownian motion under Q , defined as $\hat{W}_1 = W_1 + \int \lambda_1(y_t) dt$. Under \bar{Q} , y_t is the solution to

$$dy_t = \left[m(y_t) - \sum_{i=1}^2 \lambda_i(y_t) v_i(y_t) + \sigma_0 v_1(y_t) \right] dt + v_1(y_t) d\bar{W}_{1t} + v_2(y_t) d\bar{W}_{2t}, \tag{53}$$

where \bar{W}_1 is a \bar{Q} -Brownian motions and $\bar{W}_2 = \hat{W}_2$. In Eq. (51), $\{B(y, t)\}_{t \geq 0}$ is a collection of bond prices in a fictitious economy in which the instantaneous interest rate is $\rho(y) \equiv R(y) + \sigma_0 \lambda_1(y) - g_0$, and the risk-neutral probability measure is \bar{Q} . By Lemma 1a, for all t , $B_y(y, t) > 0$ whenever $\rho'(y) = R'(y) + \sigma_0 \lambda'_1(y) < 0$, i.e., whenever the risk-adjusted discount rates are countercyclical. Finally by Lemma 1b, for all t , $B_{yy}(y, t) < 0$ whenever $\frac{d^2}{dy^2}[m(y) + \sigma_0 v_1(y) - \sum_{i=1}^2 \lambda_i(y) v_i(y)] < 2\rho'(y)$ and $\rho''(y) > 0$, for all y . By the definition of ρ , these two inequalities are exactly those given in Conditions a.1 and a.2 of Proposition 1a. \square

To ease notation, let $\mathcal{R}(y) \equiv \text{Disc}(y) = R(y) + \sigma_0 \lambda_1(y)$, $\hat{m} \equiv m(y) - \sum_{i=1}^2 \lambda_i(y) v_i(y)$ and $\bar{m}(y) \equiv \hat{m}(y) + \sigma_0 v(y)$. One set of technical regularity conditions required in the main text is

H1: *The functions \mathcal{R} , \bar{m} , \hat{m} and v satisfy the following conditions:*

- (i) $\min_{y \in (y, \bar{y})} \mathcal{R}(y) > g_0 > 0$.
- (ii) $\lim_{y \rightarrow \underline{y}} \bar{m}(y) \geq 0$.
- (iii) *The functions $[\bar{m}(y) + v_i(y)]^{\frac{\mathcal{R}'(y)}{\mathcal{R}(y)}}$ ($i = 1, 2$), and $\|v(y)\|^2 \frac{\mathcal{R}''(y)}{\mathcal{R}(y)}$ are bounded.*
- (iv) *The functions $|\frac{d}{dy} \bar{m}(y)|$, $|\hat{m}(y) + v_i(y)|^{\frac{\mathcal{R}'(y)}{\mathcal{R}(y)}}$ ($i = 1, 2$), and $\|v(y)\|^2 \frac{\mathcal{R}''(y)}{\mathcal{R}(y)}$ are bounded.*

Condition H1-i is an integrability condition. Condition H1-ii requires that under the measure \bar{Q} introduced in Eqs. (19) and (51), y is mean reverting in a neighborhood of \underline{y} . Finally, Condition H1-iii (respectively, H1-iv) bounds the rate of explosion of $\mathcal{R}(y)$ (respectively, $\mathcal{R}'(y)$) to infinity. I now prove Part (b) of Proposition 1.

Proof of Proposition 1 (Part (b)). By assumption, $\mathcal{R}(y)$ is strictly positive. Hence $\mathcal{R}(y_t) = \mathcal{R}(y_0) w_t(\alpha, \beta)$, $w_t(\alpha, \beta) \equiv \exp\{\int_0^t [\alpha(y_s) - \frac{1}{2}(\beta_1(y_s)^2 + \beta_2(y_s)^2)] ds + \int_0^t \sum_{i=1,2} \beta_i(y_s) d\bar{W}_{is}\}$, where \bar{W}_i ($i = 1, 2$) are \bar{Q} -Brownian motions and

$$\alpha(y) \equiv \bar{m}(y) \frac{\mathcal{R}'(y)}{\mathcal{R}(y)} + \frac{1}{2} \|v(y)\|^2 \frac{\mathcal{R}''(y)}{\mathcal{R}(y)} \quad \text{and} \tag{54}$$

$$\beta_i(y) \equiv v_i(y) \frac{\mathcal{R}'(y)}{\mathcal{R}(y)}, \quad i = 1, 2. \tag{55}$$

By Eq. (51), the price–dividend ratio $p(y)$ is the solution to the following differential equation:

$$0 = \frac{1}{2}\|v\|^2 p'' + (m + \sigma_0 v_1 - \lambda \cdot v)p' - (R - g_0 + \sigma_0 \lambda_1)p + 1, \tag{56}$$

where $v \equiv [v_1 v_2]$. The Feynman and Kac representation of p in Eq. (56) is

$$p(y) = \bar{\mathbb{E}} \left[\int_0^\infty e^{g_0 t - \int_0^t \mathcal{R}(y_u) du} dt \middle| y_0 = y \right] = \bar{\mathbb{E}} \left[\int_0^\infty e^{g_0 t - \mathcal{R}(y) \int_0^t w_u(\alpha, \beta) du} dt \middle| y_0 = y \right], \tag{57}$$

where $\bar{\mathbb{E}}$ denotes expectation under \bar{Q} . By assumption, $\lim_{y \rightarrow \underline{y}} \mathcal{R}(y) = \infty$ (large discounting). By H1-i, $\min_{y \in (y, \bar{y})} \mathcal{R}(y) > g_0$. Hence by dominated convergence and H1-iii, $\lim_{y \rightarrow \underline{y}} p(y) = 0$. By Eq. (56),

$$\frac{1}{2}\|v\|^2 p'' = -1 + (\mathcal{R} - g_0)p - (m + \sigma_0 v_1 - \lambda \cdot v)p'. \tag{58}$$

Now suppose that $\lim_{y \rightarrow \underline{y}} p'(y) = \infty$. This implies that $\lim_{y \rightarrow \underline{y}} p''(y) < 0$. Alternatively, assume that $\lim_{y \rightarrow \underline{y}} p'(y) < \infty$. By H1-ii, $\lim_{y \rightarrow \underline{y}} (m(y) + \sigma_0 v_1(y) - (\lambda \cdot v)(y)) \geq 0$. As shown above, $\lim_{y \rightarrow \underline{y}} p(y) = 0$. Hence, Eq. (58) implies that for small y , $\frac{1}{2}\|v\|^2 p'' \leq -1 + \mathcal{R}p$. Hence, $\lim_{y \rightarrow \underline{y}} p''(y) < 0$ whenever $\lim_{y \rightarrow \underline{y}} \mathcal{R}(y)p(y) = 0$. But again $\inf_t (\mathcal{R}(y_t)) > g_0$, and the result follows by H1-iii and dominated convergence,

$$\begin{aligned} \lim_{y \rightarrow \underline{y}} \mathcal{R}(y)p(y) &= \lim_{y \rightarrow \underline{y}} \bar{\mathbb{E}} \left[\int_0^\infty e^{g_0 t} \mathcal{R}(y) e^{-\int_0^t \mathcal{R}(y_u) du} dt \middle| y_0 = y \right] \\ &= \lim_{y \rightarrow \underline{y}} \bar{\mathbb{E}} \left[\int_0^\infty e^{g_0 t} (\mathcal{R}(y) e^{-\mathcal{R}(y) \int_0^t w_u(\alpha, \beta) du}) dt \middle| y_0 = y \right] = 0. \end{aligned} \tag{59}$$

Next, suppose that $\text{Disc}(y)$ is bounded, but that $\mathcal{R}'(y) < 0$ and that $\lim_{y \rightarrow \underline{y}} \mathcal{R}(y) = -\infty$ (large asymmetry in discounting). Because $\mathcal{R}'(y)$ is strictly negative, it satisfies $\mathcal{R}'(\hat{y}_t) = \mathcal{R}'(y) w_t(\hat{\alpha}, \hat{\beta})$, where now $w_t(\hat{\alpha}, \hat{\beta}) \equiv \exp\{\int_0^t [\hat{\alpha}(y_s) - \frac{1}{2}(\hat{\beta}_1(y_s)^2 + \hat{\beta}_2(y_s)^2)] ds + \int_0^t \sum_{i=1,2} \hat{\beta}_i(y_s) d\hat{W}_{is}\}$ and

$$\hat{\alpha}(y) \equiv \hat{m}(y) \frac{\mathcal{R}''(y)}{\mathcal{R}'(y)} + \frac{1}{2} \|v(y)\|^2 \frac{\mathcal{R}'''(y)}{\mathcal{R}'(y)} \quad \text{and} \tag{60}$$

$$\hat{\beta}_i(y) \equiv v_i(y) \frac{\mathcal{R}''(y)}{\mathcal{R}'(y)}, \quad i = 1, 2. \tag{61}$$

Moreover, by differentiating Eq. (58), and applying the Feynman and Kac representation to the solution of the resulting differential equation (similarly as in the proof of Lemma 1), I find that

$$p'(y) = -\mathbb{E} \left[\int_0^\infty \kappa_t \mathcal{R}'(y_t) p(y_t) dt \middle| y_0 = y \right] = -\mathcal{R}'(y) \mathbb{E} \left[\int_0^\infty \kappa_t w_t(\alpha, \beta) p(y_t) dt \middle| y_0 = y \right], \tag{62}$$

where the second line follows by the solution for $\mathcal{R}'(y)$, the process κ_t is such that $-\frac{d}{dt} \log \kappa_t = \mathcal{R}'(y_t) - g_0 - \frac{d}{dy_t} (m(y_t) + \sigma_0 v_1(y_t) - \lambda(y_t) \cdot v(y_t))$, and, finally, y_t is the solution to

$$dy_t = M(y_t) dt + v(y_t) d\hat{W}_t, \quad y_0 = y, \tag{63}$$

where $M = m + \sigma_0 v_1 - \lambda \cdot v + \frac{1}{2}(\frac{d}{dy}\|v\|^2)$. By $\lim_{y \rightarrow \underline{y}} \mathcal{R}'(y) = -\infty$ and \mathcal{R} bounded, $\lim_{y \rightarrow \underline{y}} p'(y) = \infty$, and hence, $\lim_{y \rightarrow \underline{y}} p''(y) < 0$. \square

Proof of Proposition 1 (Necessary conditions for countercyclical price sensitivity $S_p(y) = p'(y)$). By differentiating Eq. (58) twice and applying the Feynman and Kac representation to the solution of the resulting differential equation (similarly as in the proof of Lemma 1 and Proposition 1b),

$$p''(y) = \mathbb{E} \left[\int_0^\infty z_t (\mathcal{A}(y_t) p'(y_t) - \mathcal{R}''(y_t) p(y_t)) dt | y_0 = y \right], \tag{64}$$

where z_t is some strictly positive adapted process, and the function $\mathcal{A}(y)$ has been defined in Proposition 1a. Because discount rates are countercyclical, $p' > 0$ by the proof of Proposition 1a. Then suppose that p is concave (i.e., $p'' < 0$). By Eq. (64), there must be a set of values of y with strictly positive measure on which either $\mathcal{A}(y) < 0$ or $\mathcal{R}''(y) > 0$, or both. \square

Finally, I provide a proof of a claim made in Section 2.2 (in fact, a direct corollary to Lemma 1). Under risk-aversion corrections comparable to those discussed in Section 2.2, the investors' expectation of the future discount rates $\text{Disc}(y_t)$ fluctuates more in bad times than in good times.

Corollary 1. *Suppose that the discount rates are countercyclical but not necessarily asymmetric, i.e., $\frac{d^2}{dy^2} \text{Disc}(y) < 0$ and $\frac{d^2}{dy^2} \text{Disc}(y) \geq 0$, and that, under the probability \bar{Q} in Eq. (19), the expectation of the instantaneous changes in y_t is concave in the current state y , i.e., $\frac{d^2}{dy^2} \bar{m}(y) < 0$. Then the expectation under \bar{Q} , $\bar{\mathbb{E}}[\text{Disc}(y_t) | y_0 = y]$ is decreasing and convex in y , for all t .*

Proof. Follows by Lemma 1, after setting $\psi(y) \equiv \text{Disc}(y)$, $\rho(y) \equiv 0$ and $b(y) \equiv \bar{m}(y)$. \square

Proof of Proposition 2. The proof is similar to the proof of Lemma 1 and Proposition 1. For space reasons, it is sketched. By no-arbitrage, the price–dividend ratio p satisfies

$$0 = \frac{1}{2} \sigma_x^2 p_{xx} + [\gamma(\bar{x} - x) - \lambda_x \sigma_x] p_x + \frac{1}{2} \sigma_s^2 p_{ss} + [\kappa(\bar{s} - s) - \lambda_s \sigma_s] p_s + \frac{1}{2} \sigma_q^2 p_{qq} + [\beta(\bar{q} - q) - \lambda_q \sigma_q] p_q - (R + \lambda_D \sigma_D - x) p + 1. \tag{65}$$

By differentiating the previous equation twice with respect to s and once with respect to q leaves three differential equations taking the form $L^i w^i - \rho^i w^i + h^i = 0$, where $w^1 = p_s$, $w^2 = p_{ss}$, $w^3 = p_q$, and ρ^i and h^i are some functions of x , s and q . By the same arguments used for the proof of Lemma 1, the sign of w^i is inherited by the sign of the functions h^i . The functions h^i are

$$h^1 \equiv - \frac{\partial \text{Disc}}{\partial s} p, \tag{66}$$

$$h^2 \equiv - \left[2 \frac{\partial \text{Disc}}{\partial s} + \frac{\partial^2 (\lambda_s \sigma_s)}{\partial s^2} \right] p_s - \frac{\partial^2 \text{Disc}}{\partial s^2} p, \tag{67}$$

$$h^3 \equiv - \frac{\partial \text{Disc}}{\partial q} p - \frac{\partial (\lambda_s \sigma_s)}{\partial q} p_s + \frac{1}{2} \frac{\partial \sigma_s^2}{\partial q} p_{ss}. \tag{68}$$

Part (a) of the proposition follows by the expression for h^1 and the assumption that $\frac{\partial}{\partial s} \text{Disc} > 0$. As regards Part (b), we have that $p_q < 0$ whenever $h^3 < 0$. Because $\frac{\partial}{\partial s} \text{Disc} > 0$, then $p_s < 0$. Hence, if $-\frac{\partial(\lambda_s \sigma_s)}{\partial q} > 0$, as assumed in Part (b), then $p_q < 0$ under Condition b.1. Moreover, if $-\frac{\partial(\lambda_s \sigma_s)}{\partial q} > 0$, then Condition b.1 \implies Condition b.2 because, again, $\frac{\partial}{\partial s} \text{Disc} > 0$ and hence $p_s < 0$. Finally, notice that the conditions in Eqs. (37)–(38) in the main text imply that $h^2 < 0$ and hence, $p_{ss} < 0$. \square

Interest rate and risk premia in a nonexpected utility environment [Eq. (40)]. By assumption, consumption growth is solution to Eq. (39), and expected consumption growth x_t , consumption volatility s_t , and the volatility of volatility q_t are mean-reverting processes:

$$\begin{aligned} x_{t+\Delta t} - x_t &= \gamma(\bar{x} - x_t) \Delta t + \sigma_x \varepsilon_{2,t+\Delta t} \sqrt{\Delta t}, \\ s_{t+\Delta t} - s_t &= \kappa(\bar{s} - s_t) \Delta t + \sigma_s(s_t, q_t) \varepsilon_{3,t+\Delta t} \sqrt{\Delta t}, \\ q_{t+\Delta t} - q_t &= \beta(\bar{q} - q_t) \Delta t + \sigma_q(q_t) \varepsilon_{4,t+\Delta t} \sqrt{\Delta t}, \end{aligned} \tag{69}$$

where $\varepsilon_t = [\varepsilon_{1t}, \dots, \varepsilon_{4t}]^\top$ is a vector of independent and identically distributed standard normal variables [ε_{1t} has been defined in Eq. (39)]. Let $\tilde{R}_{t+\Delta t}$ be the arithmetic return in Eq. (40),

$$\tilde{R}_{t+\Delta t} \equiv \frac{P_{t+\Delta t} + D_{t+\Delta t} \Delta t - P_t}{P_t} = \mathcal{E}_t \Delta t + \text{Vol}_t \cdot \varepsilon_{t+\Delta t} \sqrt{\Delta t}, \tag{70}$$

where \mathcal{E}_t are the expected returns, and Vol_t is the vector of return volatilities. (\mathcal{E}_t and Vol_t are pinned down below.) The log-pricing kernel in Eq. (40) satisfies

$$\log \xi_{t+\Delta t} - \log \xi_t = -\theta \rho \Delta t - \frac{\theta}{\psi} \log \left(\frac{D_{t+\Delta t}}{D_t} \right) + (\theta - 1) \log(1 + \tilde{R}_{t+\Delta t}), \tag{71}$$

where

$$\log(1 + \tilde{R}_{t+\Delta t}) = \tilde{R}_{t+\Delta t} - \frac{1}{2} \|\text{Vol}_t \cdot \varepsilon_{t+\Delta t}\|^2 \Delta t + O_p(\Delta t^{\frac{3}{2}}). \tag{72}$$

Taking the limit $\Delta t \rightarrow 0$, and applying the Itô's multiplication rule,

$$d \log \xi_t = -\theta \rho dt - \frac{\theta}{\psi} d \log D_t + (\theta - 1) \left[\frac{dP_t + D_t dt}{P_t} - \frac{1}{2} \|\text{Vol}_t\|^2 dt \right], \tag{73}$$

where D_t is as in Eq. (31). For small Δt , the return in Eq. (70) satisfies

$$\frac{dP_t + D_t dt}{P_t} = \mathcal{E}_t dt + \text{Vol}_t dW_t, \tag{74}$$

where \mathcal{E} is as in Eq. (35) and, by Itô's lemma applied to the price function $P(D, x, s, q) = D \cdot p(x, s, q)$, the vector $\text{Vol} = [\text{Vol}_1, \dots, \text{Vol}_4]$ is as in Eq. (34). Therefore,

by Itô's lemma,

$$\frac{d\xi_t}{\xi_t} = \left[-\theta\rho - \frac{\theta}{\psi} \frac{dE(\log D_t)}{dt} + (\theta - 1)\mathcal{E}_t - \frac{1}{2}(\theta - 1)\|\text{Vol}_t\|^2 + \frac{1}{2}\eta^2\sigma_{D,t}^2 + \frac{1}{2}(1 - \theta)^2\|\text{Vol}_t^*\|^2 \right] dt - \frac{\theta}{\psi}\sigma_{D,t}dW_{1,t} + (\theta - 1)\text{Vol}_t dW_t, \quad (75)$$

where $\sigma_{D_t} = \sigma_D(s_t)$ and $\text{Vol}^* = [\text{Vol}_2, \text{Vol}_3, \text{Vol}_4]$. The expressions for the risk premia λ_i in the main text follow immediately. The solution for the interest rate is obtained by plugging Eq. (35) into the drift of the previous equation, by identifying the drift in Eq. (33), and by rearranging terms. (Detailed computations are available upon request.)

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