A Theory of Debt Accumulation and Deficit Cycles*

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Abstract

This paper introduces a tractable model of sovereign debt where governments cannot default strategically, but face intertemporal tradeoffs between (i) preferring more primary deficits to less and (ii) avoiding costly defaults. Governments run deficits when debt and, then, the marginal costs of increasing debt are low. However, after an extended period of debt accumulation, default probabilities begin to rise quickly, and so do the marginal costs of running debt. Eventually, debt reaches a critical level relative to the size of the economy, a fiscal tipping point, after which debt accumulation stops, with governments cycling between deficits and surpluses, until perhaps a time of default. The main conclusions are that (i) fiscal tipping points typically occur when distance-to-default is between 10% and 20%; (ii) tipping points are pushed back in a stable macroeconomic environment, such that default premiums are higher in countries that implement austerity earlier and remain positive even when exogenous risk is very small (two "volatility paradoxes"); (iii) liquidity conditions and fiscal reforms may affect default probabilities in an ambiguous way; (iv) fiscal austerity may arrive too late: "debt intolerance" arises around the fiscal tipping point.

Keywords: government debt; default; fiscal tipping points; austerity; deficit cycles, volatility paradox.

JEL: G01; G15; G38; E43; E44; E61.

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1. Introduction

Our 2010 paper found that, over the long term, growth is about 1 percentage point lower when debt is 90 percent or more of gross domestic product.

Carmen M. Reinhart and Kenneth S. Rogoff, April 26, 2013, The New York Times

Why do nations accumulate debt? Governments may represent citizens who value either high expenses or low taxes, or both. For example, the fiscal illusion doctrine, advocated long time ago (see, e.g., the discussion of Buchanan and Wagner, 1977), explains conspicuous trends in government debt based on the assumption that electors have limited knowledge of the intertemporal implications of their preferences for deficits. In fact, there is strong evidence in support of subsequent political economy theories,[1] by which governments face fully rational voters and run fiscal policies that do not necessarily maximize the welfare of a representative consumer. To illustrate, a policymaker in office may not fully internalize the debt burden while facing a probability of not being re-elected, or in the presence of political polarization. The result is a deficit bias, a conclusion known at least since Alesina and Tabellini (1990). In the last century, promoting fiscal deficits has been a policy pursued by a variety of governments. Countries such as Italy or Belgium would amass a level of debt that was unimaginable at the time of the debt stabilization programs that followed World War II. In the U.S., national debt has reached levels that feed liquid markets for derivatives and other instruments based thereon.

This paper develops a model in which governments' preferences do indeed come in the form of preferring more primary deficits to less. A government faces the following intertemporal tradeoffs. By running deficits, it achieves the goal of reaping its short-term benefits such as, say, the immediate utility of satisfying the citizens it represents. But running deficits for too long may lead to costly default and, then, limited access to financial markets, implying some inability to run further primary deficits. Therefore, a government may optimally undergo periods of "austerity" in an attempt to avoid the costs of failing to ensure debt sustainability. We provide conditions under which these tradeoffs arise under an incentive-compatible mandate that governments receive by the citizens. Our model predicts that, when debt is small compared to the size of the economy, governments accumulate debt for some period. The length of this period increases with (i) the growth rate of the economy, (ii) government short-sightedness, (iii) macroeconomic stability, (iv) the expected time defaulted governments may re-gain access to capital markets, (v) the expected severity of austerity measures, (vi) debt market liquidity.

These properties may help provide new explanations for the high levels of debt accumulated in many countries since World War II (see, e.g., IMF, 2019). They rely on the assumption that governments prefer more deficits than less, as explained. Yet, and unlike in the fiscal illusion doctrine,

[1] See, e.g., Alesina, Roubini and Cohen (1997) and Drazen (2000) for early surveys of these theories and the evidence. Alesina and Passalacqua (2016) provide a relatively more recent review.

policymakers are concerned about costly default. In fact, as explained, the model predicts that there is a "fiscal tipping point," that is, a triggering value for the debt-to-GDP ratio, beyond which governments undergo a regime switch, running surpluses and contributing to improved debt statistics. However, once debt statistics improve, governments may begin another season of deficits, and this process may continue for long time. Naturally, the literature contains well-known explanations for prolonged episodes of debt accumulation and subsequent stabilization programs; one of the most known instances relies on the war-of-attrition model of Alesina and Drazen (1991). Our model contributes to this literature by providing a different mechanism for such programs: the trade-off between the short-term benefits of primary deficits and avoiding costly defaults.

In our model, governments enter into austerity when the marginal costs of running debt become too large. These costs are endogenous in the model. Thus, the model provides a rationale for the empirical evidence in the literature stemming from Bohn's (1998, 2008) Fiscal Reaction Function: a positive response of the primary balance to outstanding debt (see, e.g., D'Erasmo, Mendoza, and Zhang, 2016). For example, Table 1 provides evidence that the probability of implementing austerity plans increases in periods with unusually high debt, in line with the model predictions.

α	A_α	B_α	R^2	Tip
95%	0.088 (7.21)	0.842 (9.06)	0.27	3%
90%	0.178 (7.149)	0.743 (3.72)	0.06	14%
85%	0.168 (6.922)	0.679 (3.41)	0.05	14%
80%	0.220 (8.01)	0.671 (2.86)	0.03	20%
75%	0.211 (7.82)	0.605 (2.59)	0.01	20%
70%	0.280 (9.30)	0.420 (1.60)	0.01	28%

Table 1. Estimation of an asymmetric Fiscal Reaction Function for the U.S. The table reports estimates of the coefficients A_α and B_α (with t-stats in parenthesis) and R^2 in the following regression: $Tip_t(\alpha) = A_\alpha + B_\alpha(\delta_t - Q_{\delta,t}(\alpha)) + e_t(\alpha)$, where $Tip_t(\alpha) \equiv \mathbb{I}_{S_t > Q_{S,t}(\alpha)}$, $\mathbb{I}_{S_t > Q_{S,t}(\alpha)}$ is an indicator that takes a value equal to one when S_t, the primary surplus over GDP, is larger than $Q_{S,t}(\alpha)$, δ_t is the debt-to-GDP ratio, $e_t(\alpha)$ is an error term and, finally, $Q_{X,t}(\alpha)$ denote the α-quintile of a variable X at time-t, estimated through the previous ten years of data. Data are yearly and cover the sample from 1792 to 2012, for a total of 221 observations. The column labeled "Tip" reports the fraction of time $Tip_t(\alpha) = 1$ in this sample.

When do fiscal tipping points occur? In the model, default may occur due to exogenously given liquidity crises—a limited market capacity to absorb new debt. We derive a bound on the debt-to-GDP ratio, such that default occurs once this bound is hit. The model predicts that the tipping

point, $\hat{\delta}$, and the default boundary, $\bar{\delta}$, satisfy $\hat{\delta} = C\bar{\delta}$, for some constant C. Default may occur both in a deficit (i.e., $C > 1$) and in a surplus regime ($C < 1$). In a wide set of calibrations, we find tipping points at a level between 80% and 90% of the default boundary (i.e., C is mostly in the range of $[0.80, 0.90]$). In other words, fiscal tipping points act as "gravity centers": debt builds up over an extended period of time, but, then, cycles around the tipping point until perhaps a time of default. Debt is mean-reverting due to the governments' responses to economic conditions.

In a famous paper, Reinhart and Rogoff (2010) argue that there are values of debt-to-GDP ratios that trigger severe economic downturns. Our tipping points are expressed in terms of distance-to-default (i.e., $1 - C$), but share similarities with Reinhart and Rogoff triggering figures. In our model, governments implement austerity plans only once they have determined that the probability of defaulting has reached a level that is unacceptably high (from their viewpoint): they understand that default is imminent. It may be possible that a sharp decrease in output may, then, occur due to deteriorated market expectations and financing conditions. Our model, however, is silent regarding the forces underlying these linkages.[2] Our main point is to identify *when* governments implement austerity, and our analysis suggests that austerity may arrive "too late": governments with preferences for deficits enter into austerity only when debt statistics are quite deteriorated even from their own viewpoint (in the calibrations of the model, when distance-to-default is between 10% and 20%).

Tipping points are model predictions that may obviously offer a simplified representation of actual governments' behavior. Yet they provide a notion of a *structural* fiscal reaction function, which we may use for implementing a number of experiments that are not subject to Lucas' critique. Thus, for example, our model provides predictions regarding how default probabilities change following certain fiscal reforms (see below), which fully incorporates governments' behavior (i.e., a new tipping point) consistent with these reforms. We then study the model implications on the cost of capital that governments face to compensate investors for the risk of default. The spread of this cost against the safe interest rate (in short, the "spread") displays a number of properties. Remarkably, it rises very sharply around the fiscal tipping point, a property linked to the previous explanation that austerity may "arrive too late." Furthermore, it increases with government short-sightedness and the probability of re-entry into capital markets after default. Under conditions, the spread may temporarily decrease upon a fiscal expansion that is conducive to growth; one amongst such conditions is that the expansion is made when debt statistics are sufficiently away from the default boundaries. Finally, we find that the spread increases with macroeconomic volatility, thereby rationalizing the empirical evidence provided by Hilscher and Nosbusch (2010).

These properties result from different forces, some mechanical and others related to governments' behavior. Consider, for example, macroeconomic volatility. In our model, austerity results from governments' concerns for costly default. Therefore, governments that face relatively more volatile economies tend to act more prudently, by lowering their fiscal tipping points (property (iii) men-

[2] In fact, in a seminal paper, Giavazzi and Pagano (1990) suggest that fiscal austerity may be expansionary, due perhaps to the expectation that future economic performance will not be engulfed with critically high taxation.

tioned at the beginning of the introduction). We find that this effect may be very strong and lead countries with lower volatility to accumulate more debt in the future and, for this reason, to still experience a sizeable default probability. While this property may appear surprising, it parallels similar mechanisms known as the "volatility paradox," which have been put forward in other contexts (Brunnermeier and Sannikov, 2014): risk may build up precisely in times with low volatility. Our model predicts that (i) spreads are still increasing in volatility, as explained, but that (ii) they are still positive in economies with quite low volatility.

Similar mechanisms explain the relation between liquidity conditions and the size of debt. Property (vi) implies that liquidity support (by, say a central bank) provides governments with incentives to postpone the time for austerity; thus, while this support may help maintain spreads low in bad times, the moral hazard it creates is responsible for debt and spreads to increase sizably in the future. Debt and default probabilities may be much lower when governments know they have to face market discipline in the form of higher costs for borrowing in the first place.

The paper is organized as follows. The next section provides links of this paper to the literature. Section 3 develops model and extensions. Section 4 explains the model determinants of the fiscal tipping points. Section 5 determines credit spreads predicted by the model. Section 6 concludes. Three appendixes contain technical details and model extensions omitted from the main text. An Internet Appendix contains additional technical details, extensions and calculations.

2. Relations to the literature

The standard approach to modeling sovereign debt and default relies on the assumption that a benevolent planner maximizes the welfare of a representative agent. In a seminal paper, Eaton and Gersovitz (1981) consider a model in which governments rebate to consumers the proceeds from their international credit operations, and are able to strategically default. Aguiar and Gopinath (2006) and Arellano (2008) develop the first extensions of this model for the purpose of producing quantitative predictions for small open economies. Aguiar, Chatterjee, Cole, and Stangebye (2016) provide a review of the large and ensuing literature.

In our model, governments are not able to decide on default timing. Moreover, they aim to maximize future deficits, not the welfare of a representative agent. However, the tradeoffs governments face may result from an incentive-compatible mandate from the citizens, as explained. Thus, by focussing on governments' incentives on public spending, our paper comes close to the political economy literature mentioned in the introduction. Note that this literature is only beginning to explore how political economy explanations of governments' behavior affect default probabilities and other asset prices properties. To date, Andreasan, Sandleris and Van der Ghote (2019) consider a model built around the lines of Eaton & Gersovitz model of strategic default; in their model, governments choose to default when they do not receive electoral support (i.e., not only when they lack sufficient resources), and default probabilities increase with income inequality and regressive taxes. By contrast,

default occurs exogenously in our model, and governments' interests are always aligned to citizens'.[3] Furthermore, we focus on the interplay of deficit biases and fiscal tipping points, and the implications of this interplay on credit spreads. Models with strategic default typically predict that default occurs while governments run primary deficits. Our model suggests a much stronger role for fiscal tipping points and austerity in coping with debt sustainability.

We consider default events determined by limited market capacity to absorb new debt. There is a large body of the literature dedicated to modeling and estimating sustainable levels of debt (see, e.g., the survey of D'Erasmo, Mendoza, and Zhang, 2016), i.e., "fiscal limits." Our model can actually be calibrated based on some of these alternative proposals, an exercise undertaken in the Internet Appendix. Note that the previous literature also contains models by which sovereign default links to a number of notions of fiscal limits (see, e.g., Bi, 2012; Bi and Traum, 2012; or Corsetti, Kuester, Meier, and Müller, 2013[4]); in these models, fiscal reaction functions are exogenous. Our model departs from this literature as its fiscal tipping points are endogenous. As we anticipated in the Introduction, this property enables us to analyze the effects of fiscal reforms (such as the budget size in austerity) without being exposed to the Lucas' critique—our fiscal tipping points change with a reform, with consistent implications on debt spreads.

Our model relies on a continuous-time stationary framework in which government choices originate from the solution of real option problems. The existence of fiscal tipping points parallels similar properties arising in the liquidity management and dividend policy problem introduced by Jeanblanc-Picqué and Shirayev (1995) and Radner and Shepp (1996), and extended to deal with agency problems by, amongst others, DeMarzo and Sannikov (2006) and Biais, Mariotti, Plantin and Rochet (2007) (see, e.g., Moreno-Bromberg and Rochet, 2018, for a survey of this literature). To draw an analogy, whereas, in this literature, the problem is one of a private firm managing cash distribution under liquidity constraints (or one of a potential conflict of interest between a principal and an agent), our problem is one of a government managing primary deficits under default constraints. In both cases, some agent takes some action (dividend or payment, in the current literature; fiscal surpluses, in this paper) as soon as some underlying endogenous variable hits a threshold. Furthermore, our analysis relies on discrete interventions (of a type considered by Jeanblanc-Picqué and Shirayev);[5] however, our incentive-compatible mandates (from citizens to governments) originate from a singular

[3]For completeness, we also solve our model based on governments' strategic behavior (see Section 3.7). In this model, we deal with default boundaries as in the standard literature on default and real options started by Leland (1994) reviewed in a moment, but with the added complication that governments still need to determine their fiscal tipping points. For a survey on the standard corporate finance literature on debt, default and real options, see Chapter 14 in Mele (2022). Duffie and Singleton (2003) and Lando (2004) provide early surveys of the literature on default in asset pricing and corporate finance.

[4]In these papers, fiscal limits are random. Our model can be extended to deal with random fiscal limits by assuming that defaults arrive as a Poisson processes. Unfortunately, the model is not analytically tractable under this assumption.

[5]That is, we do not treat fiscal tipping points as part of a singular stochastic control problem. To illustrate, we might have considered a model in which a government accumulates debt until it optimally decides for austerity, whereby a fiscal surplus would then occur at an infinite rate. In this case, debt would be driven by a regulated Brownian motion. But a fiscal surplus run at an infinite rate in a time of austerity does not seem to be plausible.

stochastic control problem, as in the previous corporate finance literature.

Finally, note that "trigger points" have also been proposed to explain government policy actions (see, e.g., Bertola and Drazen, 1993). Our trigger points are endogenous, however. Likewise, Veronesi and Pastor (2012, 2013) also consider the effects of government policies on asset prices and determine trigger points for governments' action: in their models, governments undertake policy changes when they estimate that their current policies have become sufficiently unfavorable for capital accumulation. Instead, in our model, fiscal tipping points arise because governments estimate that debt accumulation has grown sufficiently large relative to the likelihood of default.

3. Preferences for deficits and the cost of national debt

This section contains model assumptions (Section 3.1), optimality conditions (Sections 3.2 and 3.3), and implications regarding deficit policies and the utility costs of running debt (Section 3.4). Section 3.5 formulates specific assumptions on the timing to default. Section 3.6 considers some extensions to deal with the case of distortionary policies. Section 3.7 describe the basic assumptions of a model with endogenous default (solved in Appendix C).

3.1. Output and debt accumulation

We consider an economy in which output (i.e., GDP) growth is independent and identically distributed, and we assume that it is a Geometric Brownian motion with parameters μ and σ,

$$\frac{dy_t}{y_t} = \mu dt + \sigma dW_t,$$

where W_t is a standard Brownian motion. The government constraint is $\dot{D}_t = -S_t + iD_t$, where i is the short-term rate and S_t is government surplus. Therefore, the debt-to-GDP ratio, δ_t, is solution to

$$d\delta_t = -\left(s_t + \kappa\right)\delta_t dt - \sigma\delta_t dW_t, \tag{1}$$

where $\kappa \equiv \mu - i - \sigma^2$, and s_t is the surplus-to-debt ratio, $s_t = \frac{S_t}{D_t}$. In this section, we take the interest rate, i, as exogenously fixed. An exogenously fixed interest rate may result from a central bank support of debt purchases as well as coercions regarding international capital market movements. However, Sections 3.5.1 and 5.6 relax this assumption and describe some mechanisms by which compensation for credit risk is state-dependent. Finally, we assume that the government cannot invest in the financial market. Its only role is to decide on the amount of deficits. Therefore, assuming that $D_0 > 0$, the debt-to-GDP ratio is always positive, $\delta_t \geq 0$.

3.2. Governments' preferences and policy

Our central assumption is that governments draw short-run benefits from running primary deficits. These benefits may result from "political dividends" related to pressures exerted by the citizens they represent. To illustrate, citizens may neglect the longer term implications of a debt burden and support governments that assign high weight to deficits. Alternatively, Section 3.2.2 describes an incentive-compatible relationship by which citizens give governments a mandate to run the budget: the result is the same objective function in (2) below. However, governments also worry about a costly default. Thus, their budget policy reflects intertemporal tradeoffs between enjoying short-run benefits (high deficits) and the long-run implications of a prolonged deficit regime (costly default). Naturally, deficits can be high either due to large expenses or low taxation, but we do not model the origins and implications of the sources of deficits. Moreover, unless otherwise specified (see Section 3.6), we assume that deficits are not distortionary.

3.2.1. Deficit ratio targets

Quite often are governments' scopes referenced to quantities of interest such as the Gross National Product or national debt. For example, the soundness of budget policies of E.U. member states is benchmarked to deficit-to-GDP ratios (e.g., Art. 126 of the E.U. Treaty). In this paper, we assume that governments benefit from these types of *ratios* (not levels) and, in particular, from deficit-to-*debt* ratios: governments draw utility from their primary deficits relative to the existing stock of debt. By Eq. (1), we shall see, it is as if they controlled the growth rate of the debt-to-GDP ratio, δ_t, for a given κ.[6] These assumptions enable one to formulate a stationary, Markov problem, and lead to predictions that can be confronted with common policy discussions. Note that the deficit-to-GDP ratio, $s_{y,t} \equiv s_t \delta_t$, could be an alternative target. However, formulating governments' utility functions in terms of s_t simplifies Bellman equations and allows us to calculate value functions in closed-form.

Thus, we assume that a government seeks to maximize deficits for any given level of debt. Formally, it minimizes the expected discounted value of future surplus-to-debt ratios,

$$V(\delta_t) = \inf_{s_u \in [s^1, s^2]} E_t \left[\int_t^\infty e^{-\rho(u-t)} s_u du \right], \tag{2}$$

under the debt accumulation constraint in Eq. (1).[7] The time preference parameter ρ determines how myopic the government is. If, say, ρ is high, the government predominantly cares about upcoming budget outcomes. This impatience may link to pressures related to political competitiveness. For example, assume that, at any instant of time, the government faces a joint probability of facing a

[6] In models with endogenous interest rates (see Section 3.5 and 5.6), capital market conditions (and, hence, κ) are also endogenous.

[7] The Internet Appendix D contains an extension of the model that accounts for a cyclical component of the primary surplus. The extension is consistent with the empirical literature (see Cochrane, 2019), but the model predictions in the main paper remain the same. Section 3.2.2 also relies on the model in the Internet Appendix D.

snap election and losing that election equal to a constant p. The government value function satisfies $LV + s - kV + p(\underline{V} - V) = 0$, where L denotes the infinitesimal generator for the diffusion (1), k is the discount rate and $\underline{V}$ is the value resulting when the government loses the snap election. Assuming that $\underline{V}$ is constant leads to Eq. (2), with $\rho = k + p$.[8] Finally, the two constants, s^1 and s^2, represent bounds on the government feasible actions. To illustrate, international compacts may imply constraints on the extent of both deficits and surpluses. We now describe a mechanism that leads to the objective function described so far.

3.2.2. Incentive-compatible plans

This section considers a relationship between governments and citizens, which leads to the same optimality conditions for the problem in (2). Its details parallel agency problems in the corporate finance literature reviewed in Section 2. We assume that citizens benefit from the services covered by the budget, but are not able to perform these services. They, then, delegate a government while understanding that accumulation of debt may lead to costly defaults. We assume that, in the absence of any agency issues, governments can run a *cumulative* primary surplus equal to $\mathcal{S}_t$, where

$$d\mathcal{S}_t = S_t dt + \psi D_t \left(\frac{dy_t}{y_t} - E\left(\frac{dy_t}{y_t} \right) \right), \tag{3}$$

for some $\psi \geq 0$. Thus, the primary surplus now includes a cyclical component. The assumption that this component scales up with debt is not crucial, although analytically convenient. It implies that the debt-to-GDP ratio dynamics are still as in (1), but with $\sigma(1 + \psi)$ replacing σ.

However, governments may benefit from running a cumulative primary surplus $d\hat{\mathcal{S}}_t \leq d\mathcal{S}_t$, where the wasting gap, $d\mathcal{S}_t - d\hat{\mathcal{S}}_t$, can result from political embezzlement and/or management inefficiencies. We assume that this gap lead to benefits equal to $dB_t = \frac{\lambda}{D_t}(d\mathcal{S}_t - d\hat{\mathcal{S}}_t)$. Citizens do not observe $\hat{\mathcal{S}}_t$. However, they may incentivize the government to behave, $d\mathcal{S}_t = d\hat{\mathcal{S}}_t$, by offering some fixed compensation (normalized to zero) and a state contingent compensation that is set in proportion of existing debt, $\frac{dG_t}{D_t}$ say. Thus, the government instantaneous benefits amount to $\frac{d\hat{G}_t}{D_t} = \frac{dG_t}{D_t} + dB_t$, and the continuation utility is

$$q_t = E_t \left[\int_t^\tau e^{-\chi(u-t)} \frac{d\hat{G}_u}{D_u} + e^{-\chi(\tau - t)} R \right],$$

where $\chi > 0$ is the government discount rate, and $R \equiv 0$ is the government reservation value. Citizens

[8] This stylized example may be generalized to the case of elections taking place at a predictable frequency. While similar in spirit, the resulting model would be analytically intricate.

benefit from primary deficits net of the governments' compensation, and their utility is

$$E_t\left[\int_t^{\tau} e^{-\rho(u-t)}\frac{-d\hat{\mathcal{S}}_u - d\hat{G}_u}{D_u} - e^{-\rho(\tau-t)}L\right],$$

where $\rho > 0$ is a discount rate, and L is the expected cost that the citizens would have to bear at the termination of the government mandate: it can be thought of as the value of the public infrastructure that allows new governments to operate. We assume that, upon termination, new governments are immediately found that have continuation utility equal to some $q_0 > 0$. Finally, we assume that citizens are more patient than governments, i.e., $\rho < \chi$.

The citizens' objective is to maximize the expected flow of future services while incentivizing governments to behave. In Appendix A (see Lemma A.1), we show that the dynamics of the governments' continuation utility are given by

$$dq_t = \chi q_t dt - \frac{d\hat{G}_t}{D_t} + \psi\sigma\phi_t dW_t, \tag{4}$$

for some process ϕ_t, and that the mandate is incentive-compatible if $\phi_t \geq \lambda$ for all t. Thus, the citizens' value function can be expressed as

$$\mathcal{V}(q,\delta) = \inf_{\phi_t\geq\lambda, dG_t\geq 0, s_t\in[s^1,s^2]} E\left[\int_0^{\tau} e^{-\rho t}\frac{d\mathcal{S}_t + dG_t}{D_t} + e^{-\rho\tau}L\right], \tag{5}$$

subject to (3), to (4) (with $d\hat{G}_t = dG_t$), to (1) (with $\sigma(1+\psi)$ replacing σ), and to the boundary condition $\mathcal{V}(0,\delta) = L$ for all δ. In Appendix A, we show that $\phi_t = \lambda$ for all t (see Lemma A.2) and that

$$\mathcal{V}(q_t,\delta_t) = \mathcal{G}(q_t) + V(\delta_t), \tag{6}$$

where $V(\cdot)$ is identical to (2), and $\mathcal{G}(q_t)$ is a convex function (see Figure A-1). Thus, the governments' objective in (2) can be understood as an incentive-compatible mandate that the citizens give to governments for executing the budget services. In particular, note that the citizens' discount rate ρ is the same as the governments' discount rate in (2). In the remainder, we shall refer to the parameter ρ as the *governments'* impatience rate.

3.2.3. Default

If the government was exempt from a costly default, it would run as much deficit as possible. However, we assume that, due to limited market liquidity, there exists an amount of debt beyond which, government debt may not receive market support. Default occurs when primary deficits plus the cost of debt services cannot be financed through new debt issuance (see Section 3.5). Default is costly both due to direct costs and because governments could not run deficits until a random re-entry in the

financial markets after a period of exclusion. Thus, the value function satisfies the default condition, $V(\bar{\delta}) = \bar{V}$, for some $\bar{\delta}$ and $\bar{V} > 0$. Section 3.5 provides details on both the "debt limit," $\bar{\delta}$, and the value function in the default state, $\bar{V}$.

3.2.4. Optimality

We determine optimality conditions for the government problem. Consider the Bellman equation

$$0 = \inf_s \left[LV(\delta) - \rho V(\delta) + s\right] \equiv \frac{1}{2}\sigma^2\delta^2 V''(\delta) - \kappa\delta V'(\delta) - \rho V(\delta) + \inf_s \left[s\left(1 - V'(\delta)\,\delta\right)\right]. \tag{7}$$

Eq. (7) is subject to boundary conditions. One is that $V\left(\bar{\delta}\right) = \bar{V}$, as discussed; other conditions are discussed in Appendix A. Eq. (7) suggests that the government deficit strategy should be "bang-bang," that is: $s_t = s^1$ for all values of $\delta_t : V'(\delta_t)\,\delta_t < 1$ and $s_t = s^2$, otherwise. In this paper, we take $s^1 < 0$ and $s^2 > 0$. Given this assumption, and additional properties of $V(\delta)$ discussed in Section 3.3, the result is that the government would indeed run a deficit when the debt-to-GDP ratio $\delta_t < \hat{\delta}$, and a surplus otherwise, for some switching point $\hat{\delta}$.[9] For simplicity, we shall refer to deficit and surplus regimes as those regimes arising when $\delta_t < \hat{\delta}$ and $\delta_t > \hat{\delta}$. Moreover, we shall refer to the threshold, $\hat{\delta}$, as the *fiscal tipping point*.

3.3. The marginal cost of debt

Fiscal tipping points bear a natural interpretation. Note, first, that the value function, $V(\delta_t)$, is the minimized expected value of future surpluses, that is, the utility cost the government incurs when the debt-to-GDP ratio is δ_t. Below, we show that $V(\delta)$ is increasing: the closer to default, the higher the utility cost (see Proposition I). Next, suppose a government is running some primary deficit. At each point in time, it may either raise debt by some $\delta\Delta$ or immediately run a surplus of the same amount. The first alternative implies an increase in its utility costs equal to $V(\delta + \delta\Delta) - V(\delta)$; the second costs just $\delta\Delta$. Therefore, the government will keep on raising debt as long as

$$V(\delta + \delta\Delta) - V(\delta) < \delta\Delta.$$

Taking the limits of the previous condition for $\Delta \longrightarrow 0$ leaves $\mathcal{V}(\delta) \equiv V'(\delta)\,\delta < 1$, consistent with Eq. (7). That is, the government runs deficits when $\mathcal{V}(\delta)$, the marginal cost of raising debt, is less than the marginal cost of a primary surplus. In Appendix A, we show that $\mathcal{V}(\delta)$ is strictly increasing. Thus, fiscal tipping points are values of δ such that the marginal costs of raising debt and entering into austerity are the same.

[9] That is, the solution to Eq. (7) is found by determining a value for δ that triggers a switch in the equation satisfied by $V(\delta)$. These types of problems are known as Stefan problems (see, e.g., Rubenstein, 1971).

3.4. Deficit cycles

Default is costly (see Section 3.5). The next proposition provides a general expression for the government utility costs, which applies independent of the assumptions on the costs of defaulting:

Proposition I. (Fiscal tipping point and government utility costs). *There exists a threshold value of the debt-to-GDP ratio $\hat{\delta}$ such that the government runs a deficit $s_t = s^1$ for all $\delta_t < \hat{\delta}$, and a surplus $s_t = s^2$ for all $\delta_t > \hat{\delta}$. The utility costs satisfy $V(\delta_t) = V_{\mathcal{D}}(\delta_t)\mathbf{1}_{\delta_t<\hat{\delta}} + V_{\mathcal{S}}(\delta_t)\mathbf{1}_{\delta_t>\hat{\delta}}$, where*

$$V_{\mathcal{D}}(\delta) = \frac{s^1}{\rho} + A_{\mathcal{D}2}\delta^{m_{\mathcal{D}2}}, \quad V_{\mathcal{S}}(\delta) = \frac{s^2}{\rho} + A_{\mathcal{S}1}\delta^{m_{\mathcal{S}1}} + A_{\mathcal{S}2}\delta^{m_{\mathcal{S}2}},$$

for some constants $\hat{\delta}$, $m_{\mathcal{D}2} > 0$, $m_{\mathcal{S}1} < 0$, $m_{\mathcal{S}2} > 0$, $A_{\mathcal{D}2} > 0$, $A_{\mathcal{S}1} < 0$, $A_{\mathcal{S}2} > 0$ provided in Appendix A. The threshold $\hat{\delta}$ decreases with the utility costs at default, $\bar{V}$.

The threshold $\hat{\delta}$ is the fiscal tipping point, and, as explained in Section 3.2, the utility costs at default are $\bar{V} = V(\bar{\delta})$, where $\bar{\delta}$ is the default boundary. According to parameter values, default may occur either in the deficit regime (in which case $\hat{\delta} > \bar{\delta}$) or in the surplus regime (i.e., when $\hat{\delta} < \bar{\delta}$). Figure 1 depicts the utility costs of Proposition 1 when default occurs in the deficit regime and default costs are specified as in Sections 3.5.1 and 3.5.3. Note that the utility costs are increasing and convex in δ_t, a property discussed below.

There are two regions separated by the fiscal tipping point, $\hat{\delta}$. The region for which $\delta_t < \hat{\delta}$ identifies the deficit regime, i.e., the region where a government runs a primary deficit. In this region, the utility costs are small and relatively insensitive to changes in debt for a while. Governments run a deficit policy precisely because the marginal costs of increasing debt are also small, $V'(\delta_t)\delta_t < 1$, as explained in Section 3.3. Note that a primary deficit policy may last for an extended period of time. For example, given the parameter values in Figure 1, one has that $E_t\left(\frac{\delta_T}{\delta_t}\middle|\max_{u\in[t,T]}\delta_u \leq \hat{\delta}\right) = e^{0.0425(T-t)}$; that is, during a deficit regime, we expect that a debt-to-GDP ratio could triple from, say, 20% to 60% in more than 25 years.

When the debt-to-GDP ratio becomes sufficiently high, government utility costs grow at a progressively higher pace: as δ_t increases, so does the marginal cost of running additional deficits. When δ_t hits $\hat{\delta}$, governments implement a switch in regime, engaging into an austerity policy, the second region in Figure 1. In this region, governments run surpluses in an attempt to avoid costly default. Naturally, once δ_t hits $\hat{\delta}$ from above (the surplus region), another season of deficits begins; governments may then cycle through primary surpluses and deficits, depending on which regime in Figure 1 δ_t belongs to.

While utility costs are increasing and convex in the benchmark parametrization of Figure 1, utility costs may be concave in δ_t for other parameter constellations. For example, they are concave in the version of the model with strategic default (see Section 3.6). However, as explained in Section 3.3,

it *always* holds that the marginal costs of debt are increasing in δ_t: the government policy is to run deficits when both utility costs and marginal costs of increasing debt are small, and to switch to surplus otherwise. Section 4 studies how the tipping points change under different assumptions on the model parameters. We now turn to explain how $\bar{\delta}$ is determined in Figure 1.

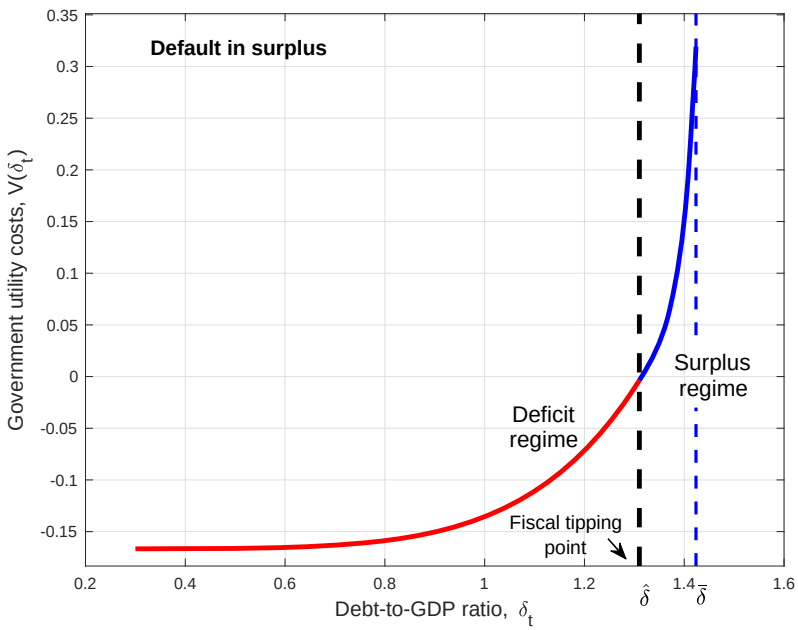

FIGURE 1: GOVERNMENT UTILITY COSTS. This picture depicts the government utility costs, $V(\delta_t)$ in Eq. (2), against the debt-to-GDP ratio, δ_t, with parameters fixed as follows:

s^1	s^2	ρ	ϑ	ϵ	γ	ξ	ℓ	$\bar{\varepsilon}$	μ	σ	i	$\bar{\iota}$
-0.05	0.05	0.30	0.30	0.05	0.80	0.20	0.10	1/12	0.03	0.05	0.02	0

The red and blue curves identify the utility costs in the regimes where a government runs a primary deficit and a primary surplus. The two regimes are separated by the fiscal tipping point, $\hat{\delta}$, identified by the tick dashed vertical line. The default boundary, $\bar{\delta}$, is identified by the thin dashed vertical line, and is obtained through Eq. (8) in Section 3.5. The parameters ℓ, ϑ, ϵ, γ, ξ, $\bar{\varepsilon}$, $\bar{\iota}$ are introduced in Section 3.5.

3.5. Debt limits

Whilst governments cycle through primary deficits and surpluses, investors may not be able to absorb any amount of new debt. In Section 3.5.1, we define a debt limit as the level of debt $\bar{\delta}$, beyond which, new deficits may not be re-financed due to current or potential liquidity crises. Default occurs at such $\bar{\delta}$. Section 3.5.2 contains a succinct discussion of alternative measures of debt limits, some of which we experiment with in the Internet Appendix B. Finally, we assume that, after default, the

government enters into a period of financial exclusion, but may re-gain access to financial markets with some exogenously given probability. Section 3.5.3 provides details on the utility costs incurred at default as a result of these assumptions.

3.5.1. Short-term spreads and a debt limit

We assume that, at any point in time, t, a large number of investors compete to supply funds at an instantaneous rate equal to $i + \iota_t$, and for a small amount of time equal to $\bar{\varepsilon}$.[10] We assume that the credit spread, ι_t, is an increasing function of δ_t, i.e., $\iota_t \equiv \iota(\delta_t)$ and denote $\bar{\iota} \equiv \max_\delta \iota(\delta)$, for some $\bar{\iota} : |\bar{\iota}| < \infty$. This assumption captures the idea that as δ_t approaches the default boundary, there are fewer and fewer investors able to bear risks, that is, the supply of funds decreases (and, mechanically, the market clearing interest rate increases) with the risk of default. We also assume that the government may finance its total deficit through new debt issuance, provided however this additional debt is less than a fixed proportion of GDP: $(-s_t + i + \iota(\delta_t))\,\delta_t < \ell$, for some $\ell > 0$. We refer to this inequality as the government *liquidity* constraint. We assume that all investors immediately pull from the market once this limit is hit, leaving the government in default and unable to renew outstanding debt.

Investors lend at t, if the government liquidity constraint is met, but also if they estimate that the government will not face a liquidity crisis at any time before $\bar{\varepsilon}$, based on the following worst-case scenario: $\sup_{s,\iota} E_t\left(-s\delta_{t+\varepsilon} + (i+\iota)\,\delta_{t+\varepsilon}\right) < \ell$, that is, $\delta_t\left(-s^1 + i + \bar{\iota}\right) e^{-(s^1+\kappa-\bar{\iota})\varepsilon} < \ell$ for all $\varepsilon \in (0, \bar{\varepsilon})$. We term this inequality *risk-bearing capacity* constraint. We assume that the total maximum amount of deficit over debt is large enough, compared to GDP growth, i.e., $\mu - \sigma^2 < -s^1 + i + \bar{\iota}$. Under this condition, both liquidity and risk-bearing capacity constraints hold when

$$\delta_t < \bar{\delta} \equiv \frac{\ell e^{(s^1+\kappa-\bar{\iota})\bar{\varepsilon}}}{i + \bar{\iota} - s^1}. \tag{8}$$

We define default time as $\tau = \inf\left\{t : \delta_t = \bar{\delta}\right\}$. Eq. (8) provides the expression for the default boundary $\bar{\delta}$ that is utilized in Figure 1, and is the benchmark for this paper. We now discuss alternative measures of a debt limit.

3.5.2. Alternative measures of debt limits

The main mechanisms of our model operate independent of the assumptions made on the default boundary. Therefore, one may consider alternative estimates of debt limits, which may affect the quantitative properties of the model. One such alternative relies on standard notions of sustainable debt, i.e., the level of outstanding debt that matches the present discounted value of the primary surpluses. D'Erasmo, Mendoza, and Zhang (2016) contain an extensive survey of a large literature

[10] This amount of time is arbitrarily small and may reflect delays that financial intermediaries experience whilst quitting a market segment. This assumption ensures that at least some investors suffer from a government default.

around this notion, its refinements and extensions. Consider, for example, the Natural Public Debt Limit, defined as the present value of the future surpluses in a state of fiscal crisis, that is, when the government is unable to borrow more and keeps its overlays to the minimum tolerable levels, viz

$$D_t \leq E_t \left[\int_t^{\infty} e^{-i(u-t)} S_u du \right] \leq y_t \frac{\bar{s}}{\bar{\imath}_{\mathrm{n}} - \bar{\mu}},$$

where $\bar{s}$ is surplus-to-GDP ratio resulting from the difference between the worst realization of revenues and the minimum overlays, and $\bar{\imath}_{\mathrm{n}}$ and $\bar{\mu}$ are the interest rate and output growth in such a fiscal crisis. That is,

$$\delta_t < \bar{\delta}_{\mathrm{n}} \equiv \frac{\bar{s}}{\bar{\imath}_{\mathrm{n}} - \bar{\mu}}. \tag{9}$$

In the Internet Appendix B, we solve the model while relying on $\bar{\delta}_{\mathrm{n}}$ as a measure of a debt limit; in Sections 4 and 5, we comment on the main implications of these assumptions on fiscal tipping points and default probabilities. Note that there are still additional notions of debt limits that may serve as estimates of the default boundary for our model. Amongst the most recent is the maximum sustainable debt (MSD) developed by Collard, Habib and Rochet (2015), which takes into account the possibility of default. In their notion, MSD-to-GDP ratios increase with the maximum primary surplus and the average growth rate of the economy, and decrease with growth rate volatility. Of course, it may be possible to employ this notion as an alternative input needed to compute the solution of our model. We leave the task of calibrating the model to alternative definitions of debt limits to future empirical investigations.

3.5.3. Re-entry and default costs

After default, the government loses access to financial markets and is bound to a no-deficit policy. However, it may eventually re-gain access to markets. We assume that the probability of re-entry has a constant intensity equal to ϑ, and that re-entry, if any, will occur with a debt level equal to a fraction γ of the debt at default. We assume that γ is small enough to guarantee that re-entry occurs at the deficit regime, a simplifying albeit not crucial assumption.

Government default entails two types of utility costs. First, the government incurs a one-time utility cost that varies proportionally with the debt-to-GDP ratio at the time of default, $\xi\bar{\delta}$, for some positive constant ξ. This cost reflects the burden the government bears while dealing with its bankruptcy, such as litigation costs, international stigma, or loss in popularity related to trade embargoes. For example, Panizza and Borensztein (2009) explain that the political consequences of default are quite significant for incumbent governments, and that these costs might be even more severe than the direct, likely short-lived, economic costs related to financial exclusion.

Second, during the exclusion period, the government experiences a utility cost equal to $\epsilon > 0$ per unit of time, however small this may be. These costs may reflect the burden inherent in building up the new image needed to re-entry the markets: for example, ϵ may be a small surplus-over-debt

at default, a surplus that may be stored and distributed to creditors. Note that, in the literature, the typical costs associated with financial exclusion regard losses in real economic aggregates (see, e.g., Aguiar, Chatterjee, Cole, and Stangebye; 2016). Our costs for financial exclusion are, instead, consistent with our focus on governments' preferences for deficits: defaulting entails an inability to run deficits for a while, which results in an utility loss $\epsilon > 0$ per unit of time.

In Appendix A, we show that

$$\bar{V} = \lim_{\delta \to \bar{\delta}} \mathcal{C}(\delta), \quad \mathcal{C}(\delta) \equiv \underbrace{\frac{\epsilon}{\rho + \vartheta} + \frac{\vartheta}{\rho + \vartheta} V(\gamma\delta)}_{\equiv \mathcal{C}_d(\delta)} + \xi\delta, \tag{10}$$

where $\mathcal{C}_d(\bar{\delta})$ is the utility cost the government bears during the exclusion period. To summarize, default costs increase with the level of debt at default, due to re-entry mechanisms (i.e., through $\mathcal{C}_d(\bar{\delta})$) as well as to one-time utility costs (i.e., through $\xi\bar{\delta}$). The results in Figure 1 (see Section 3.4) are obtained based on this specification in Eq. (10).

3.6. Distortionary policies

While the paper considers an economy without production, we may use our model and assess how a given distortionary policy would affect fiscal tipping points and the dynamics of debt. To illustrate, suppose that growth depends on the regime: growth equals μ^j when $s_t = s^j$, for two values μ^1 and μ^2. We may make similar assumptions regarding the interest rate i. Under these assumptions, the Bellman equation (7) remains the same, but with $\kappa(\delta)$ replacing κ, where $\kappa(\delta) = \kappa^j \equiv \mu^j - i^j - \sigma^2$. Note that, in this formulation of the model, governments do not internalize the effects of their deficit policy on $\kappa(\cdot)$. We assume that $\kappa^1 > \kappa^2$, consistent with all the examples in Section 5. Under additional conditions (see Appendix B.1), the model solution remains the same as that in Propositions I, but with parameter values changing across the deficit or surplus regime.

Appendix B.2 also considers a model extension in which governments internalize the effects of their fiscal policy on $\kappa(s)$, such that the Bellman equation (7) is replaced with

$$0 = \frac{1}{2}\sigma^2\delta^2 V''(\delta) - \rho V(\delta) + \inf_s \left[s\left(1 - V'(\delta)\delta\right) - \kappa(s)\delta V'(\delta) \right]. \tag{11}$$

That is, in this extension, governments choose the fiscal regime (deficit or surplus, s^j) while taking into account the impact of their choice on growth (μ^j) and interest rates (i^j). We show that, under conditions, there now exist two tipping points, $\hat{\delta}_1$ and $\hat{\delta}_2$ say, such that governments undergo a deficit for $\delta < \hat{\delta}_1$, a surplus for $\delta > \hat{\delta}_1$, and a "balanced" regime with zero deficit for values $\delta_t \in (\hat{\delta}_1, \hat{\delta}_2)$. Government utility costs are, then,

$$V(\delta) = V_{\mathcal{D}}(\delta)\mathbf{1}_{\delta_t < \hat{\delta}_1} + V_{\mathcal{B}}(\delta)\mathbf{1}_{\delta_t \in (\hat{\delta}_1, \hat{\delta}_2)} + V_{\mathcal{S}}(\delta)\mathbf{1}_{\delta_t > \hat{\delta}_2}, \tag{12}$$

for three functions $V_{\mathcal{X}}(\delta)$, $\mathcal{X} \in (\mathcal{D}, \mathcal{B}, \mathcal{S})$ determined in the Appendix. However, Sections 5.5 and 5.6 analyze model predictions regarding the benchmark model without internalization effects.

3.7. Strategic default

This section examines a variant of the model in which governments choose the time of defaulting. We assume that, after repudiation of debt, the details of default costs and re-entry are the same as in the previous model with exogenous default. Thus, a government now minimizes future expected surplus-to-debt ratios while also timing default. Its utility costs are given by

$$V(\delta_t) = \inf_{\tau} \inf_{s_u \in [s^1, s^2]} E_t \left[\int_t^{\tau} e^{-\rho(u-t)} s_u du + e^{-\rho(\tau - t)} \mathcal{C}(\delta_\tau) \right], \tag{13}$$

where $\mathcal{C}(\delta_\tau)$, the costs of defaulting, are as in Eq. (10). We still assume that the government defaults while facing a liquidity crisis, as elaborated below.

To gain intuition on the government objectives, consider a small interval of time, Δt, over which the debt-to-GDP ratio is constant. At time-t, the government defaults when the costs of doing so, $\mathcal{C}(\delta_t)$, are less than the utility costs of going forward. Now, the imminent utility costs faced by the government are $s(\delta_t)\Delta t$, such that V satisfies

$$V(\delta_t) = \min \left\{ \mathcal{C}(\delta_t), s(\delta_t)\Delta t + e^{-\rho \Delta t} E_t (V(\delta_{t+\Delta t})) \right\}, \tag{14}$$

where $s(\delta_t)$ denotes the surplus policy that minimizes the utility costs conditionally upon not having defaulted yet. The costs of default include foregone deficits, but decrease with the likelihood of emerging from financial exclusion, after which the government may start running deficits again. We want to verify that governments default when they expect to be in office during a period of austerity that makes utility costs larger than the inconvenience of defaulting and remain without access to financial markets for a random period of time.

We conjecture that the government policy is to default immediately if $\delta_t \geq \bar{\delta}_o$ or, otherwise, to continue servicing its debt. In the default region, then, $V(\delta_t) = \mathcal{C}(\delta_t)$, such that, by Eq. (14),

$$0 \leq s(\delta_t) + e^{-\rho \Delta t} \frac{E_t (V(\delta_{t+\Delta t})) - V(\delta_t)}{\Delta t} - \frac{1 - e^{-\rho \Delta t}}{\Delta t} V(\delta_t),$$

where the equality holds while the government is still servicing its debt (the continuation region), in which case $V(\delta_t) \leq \mathcal{C}(\delta_t)$. In the limit, $\Delta t \longrightarrow 0$, the previous description collapses to an optimal stopping problem in continuous time. The complication in this problem is that the policy, $s(\delta_t)$, is endogenous, consistent with the inner extremization in (13). Appendix C provides conditions and verification results the previous conjectures and the Internet Appendix C describes a few key properties of the model.

4. Fiscal tipping points

This section analyzes properties of the fiscal tipping points. We consider a normalized metric, $C \equiv \frac{\hat{\delta}}{\delta}$, interpreted as the complement to distance-to-default at a fiscal tipping point, $1 - C$. For space reasons, we only discuss the exogenous default model with debt limit of Section 3.5.1 and that with the Natural Public Debt Limit (NPDL) of Section 3.5.2 (solved in the Internet Appendix B). Internet Appendix C provides results on the strategic default model in Section 3.7. Figures 2 through 5 depict the main results. Fiscal tipping points increase with growth, μ. Intuitively, the higher economic growth, the longer governments may afford to maintain deficits without defaulting. This property is robust to alternative constellations of parameter values than those reported in this section. However, there are exceptions discussed below, which arise for relatively low values of μ.

Figure 2 (left panel) shows that the normalized tipping points, C, increase with governments' short-sightedness, i.e., with ρ. Furthermore, the right panel of Figure 2 shows that the normalized tipping points increase with the frequency of re-entries, ϑ: the higher the probability of re-entry, ϑ, the higher the incentives to postpone a switch into the surplus regime. Thus, a higher ϑ implies a higher probability of default.[11] These properties may help explain "serial defaulting": higher values of ϑ lead to higher values of C, which raises the probability of default and, hence, the frequency of default-recovery cycles.

These conclusions also hold for the model with NPDL, but are more nuanced. In the Internet Appendix B, we consider one case with large and one case with tight NPDL. The model predictions in the first case are the same as those in this section. In the second case, the model with NPDL predicts that, in economies with small growth, governments with low ρ now tend to implement austerity later than myopic governments would. If growth is negative, a patient government would even give up implementing any austerity plan, thereby defaulting in the deficit regime. The reason is that, with a low growth and a tight NPDL, governments know they are likely to default soon; however, the costs of defaulting are, now, relatively small: when ρ is low, governments attach high importance to the probability of future re-entries into financial markets. In other terms, governments "gamble for resurrection" whilst enjoying short-term benefits from being in the deficit regime. Note that these effects do not arise in the benchmark model.

How does macroeconomic volatility affect fiscal tipping points? Figure 3 does show that C decreases with macroeconomic uncertainty, σ: governments hedge against macroeconomic volatility while implementing austerity earlier in economies with higher volatility. These effects become more pronounced in the presence of more myopic governments. Interestingly, the model with NPDL predicts that, in the "tight case," governments in economies with higher volatility tend to postpone austerity for longer while being confronted with weaker economies (i.e., with small or negative μ): the reason underlying this conclusion is similar to the previous gamble-for-resurrection explanations.

[11] The model with strategic default in Section 3.7 shares the same property (see Internet Appendix C). The difference is that, in this model, the probability of re-entry only affects the fiscal tipping point; in the strategic default model, the probability of re-entry also affects the default boundary.

Section 5.4 studies the implications of these properties on long-term credit spreads. To anticipate, these spreads do increase with volatility, consistent with some empirical evidence we shall review. However, the increase may be modest simply because governments enters into austerity earlier when volatility is higher.

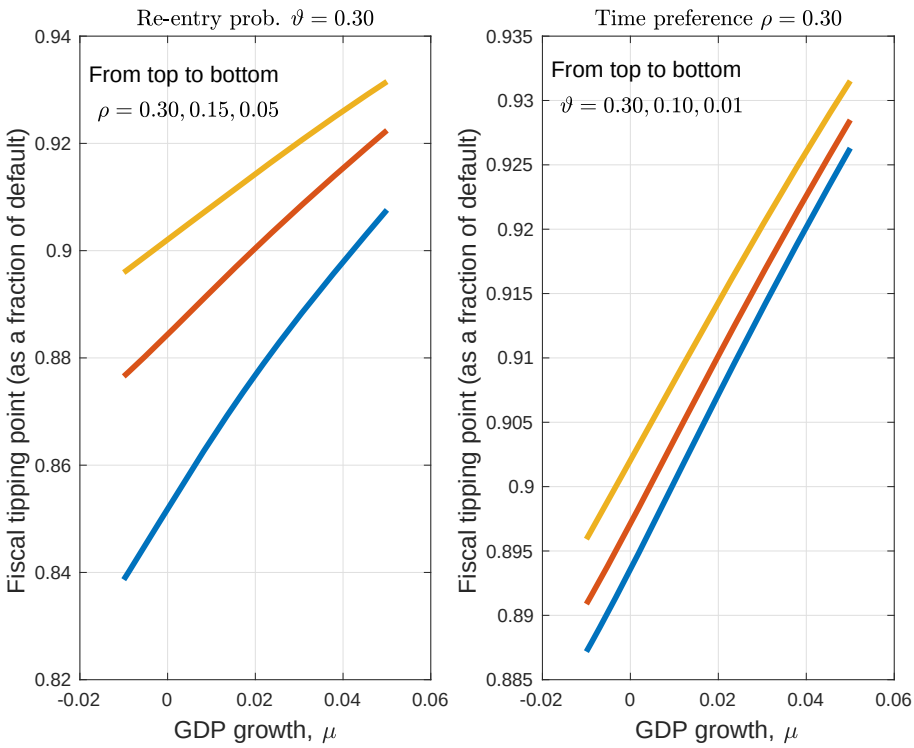

Figure 2: Fiscal tipping points, government time preferences and probability of re-entry after default. This picture depicts the debt-to-GDP ratio $\hat{\delta}$ that triggers a change in the fiscal regime (the fiscal tipping point), expressed as a fraction of the default boundary, $C = \frac{\hat{\delta}}{\delta}$, as a function of growth, μ. The left panel displays tipping points for a given probability of re-entry after default, ϑ, and varying values for the government time preference rate, ρ. The right panel displays tipping points for a given government time preference rate, and varying probabilities of re-entry after default. Remaining parameter values are as in the legend of Figure 1.

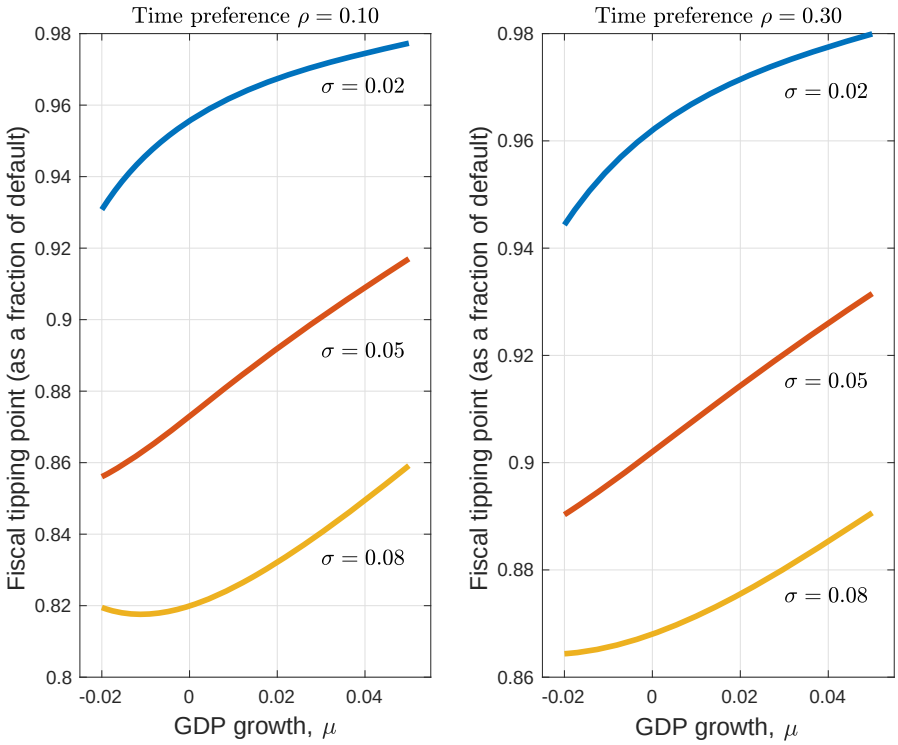

FIGURE 3: FISCAL TIPPING POINTS AND MACROECONOMIC VOLATILITY. This picture depicts the debt-to-GDP ratio $\hat{\delta}$ that triggers a change in the fiscal regime (the fiscal tipping point), expressed as a fraction of the default boundary, $C = \frac{\hat{\delta}}{\bar{\delta}}$, as a function of growth, μ. The left (resp., right) panel depicts fiscal tipping points for a government time preference rate $\rho = 0.10$ (resp., $\rho = 0.30$). Remaining parameter values are as in the legend of Figure 1.

Figure 4 plots the normalized tipping points as a function of growth, based on varying levels of the interest rate. In general, the higher the interest rate, the sooner governments enter into austerity. However, when interest rates are sufficiently high, and growth is small or negative, governments tend to postpone austerity. Intuitively, in these cases, governments gamble for resurrection: when interest rates are high and default is likely anyway, it pays to extend the deficit season as much as possible. Not surprisingly, these effects are quite strong in the model with NPDL: when debt limits are tight, this model predicts that, even in economies with moderate growth (i.e., $\mu = 2\%$), the higher the interest rate, the later governments enter into the surplus regime; for example, a moderately myopic government (i.e., with $\rho = 0.10$) would plan to default in the deficit regime when growth is moderate ($\mu = 1\%$) and interest rates are high ($i = 4\%$).

Do governments implement austerity early in the presence of large deficit sizes? Not necessarily. The first two rows of Figure 5 show that, as the size of the primary deficit increases, governments

may switch to a surplus regime *later*, at least provided growth is not too high. The reason is that defaulting becomes more likely with higher deficit sizes, such that governments gamble for resurrecting and entering the markets later and cycling again with high deficits. The model with NPDL predicts that these effects are particularly severe in the case of a tight debt limit.

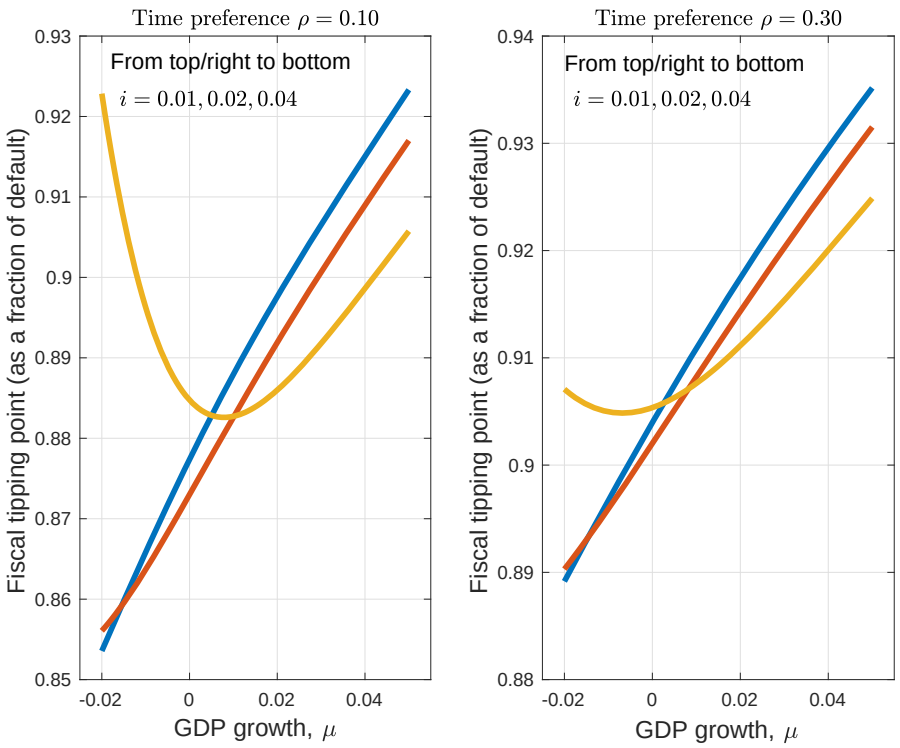

FIGURE 4: FISCAL TIPPING POINTS AND SHORT-TERM FINANCING COSTS. This picture depicts the debt-to-GDP ratio $\hat{\delta}$ that triggers a change in the fiscal regime (the fiscal tipping point), expressed as a fraction of the default boundary, $C = \frac{\hat{\delta}}{\delta}$, as a function of growth, μ. The left (resp., right) panel depicts fiscal tipping points for a government time preference $\rho = 0.10$ (resp., $\rho = 0.30$). Remaining parameter values are as in the legend of Figure 1.

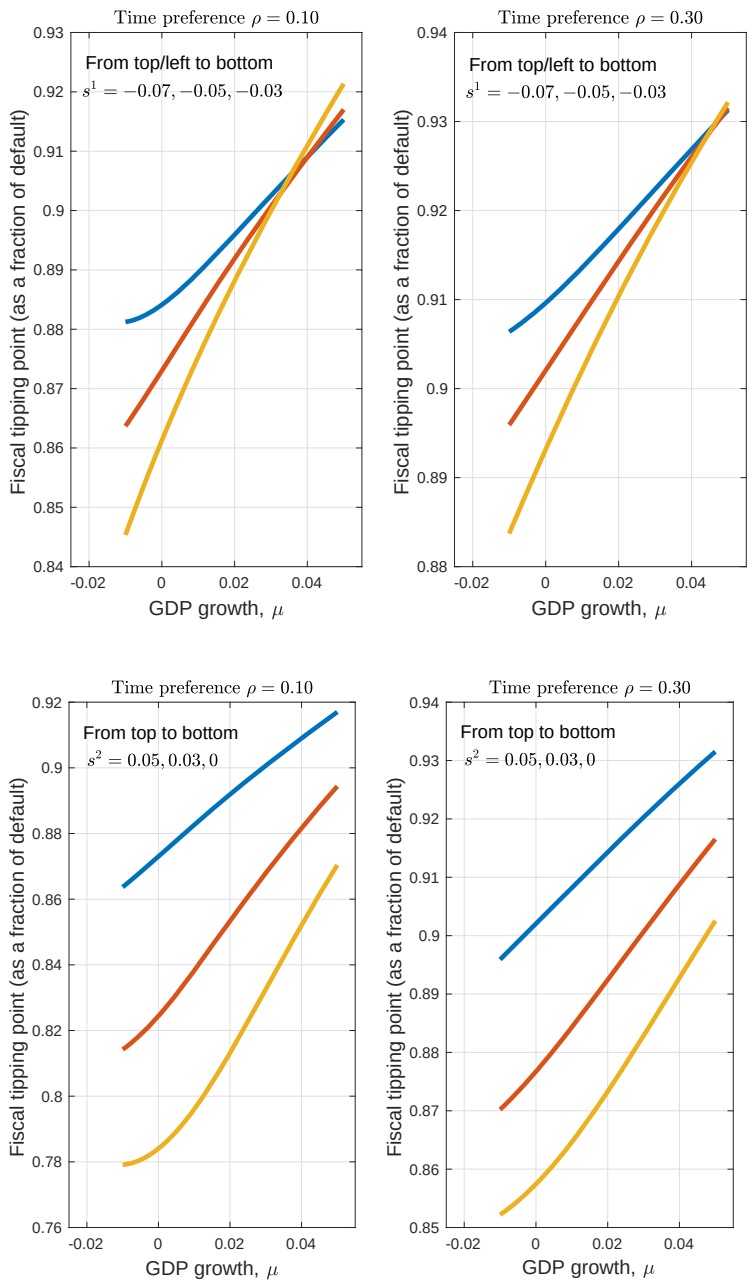

FIGURE 5: FISCAL TIPPING POINTS AND BUDGET SIZES. This picture depicts the debt-to-GDP ratio $\hat{\delta}$ that triggers a change in the fiscal regime (the fiscal tipping point), expressed as a fraction of the default boundary, $C = \frac{\hat{\delta}}{\delta}$, as a function of growth, μ. The left (resp., right) panel depicts fiscal tipping points for a government time preference rate $\rho = 0.10$ (resp., $\rho = 0.30$). Remaining parameter values are as in the legend of Figure 1.

Finally, the fiscal tipping points increase with s^2 (see the last two rows of Figure 5): as s^2 lowers, governments implement austerity earlier, due to their decreased ability to lower δ_t while in the surplus region. Results in the Internet Appendix B confirm that the same conclusions hold for the NPDL model both in the large and in the tight debt limit cases. Note that the effects of a decrease in s^2 are similar to those of an increase in the short-term spread resulting from a higher δ_t. For example, denoting with Δs^2 the primary surplus changes in Figure 5, and assuming that $\iota(\delta_t) = \mathbb{I}_{\delta_t > \hat{\delta}} \Delta s^2$, the dynamics of debt in (1) with $\left(s^2 - \Delta s^2\right) - i$ are formally the same as those in a second economy with $s^2 - (i + \iota(\delta_t))$. The difference is that, in the first experiment, the budget size is $\left(s^2 - \Delta s^2\right)$ and, in the second, is s^2 (and policies are distortionary), such that the tipping points differ in these two experiments. However, the predictions of the model with state-dependent short-term spreads are qualitatively similar to those in Figure 5 (see Appendix B.1). Section 5.5 and 5.6 analyze how deficit sizes and state-dependent short-term spreads affect default probabilities and credit spreads in this model.

5. Spreads

How do default probabilities and premiums behave contingent on given debt statistics? This section discusses the model predictions, and also provides comparative statics. To illustrate, a higher macroeconomic volatility mechanically increases default probabilities, but also leads governments to enter into austerity earlier (see Section 4). Which force dominates? More generally, the model parameters determine both fiscal tipping points and the debt dynamics that arise for a given tipping point. We analyze the ultimate effects of changes in these parameters in a number of cases that cover varying liquidity conditions, macroeconomic volatility, and assumptions underlying deficit sizes. The next section provides definitions of spreads and their connections with default probabilities, and additional details; Section 5.2 studies the behavior of spreads around fiscal tipping points; Sections 5.3 through 5.5 analyze how spreads are affected by liquidity conditions, macroeconomic volatility and budget expansions; finally, Section 5.6 studies feedback loops arising through state-dependent short-term spreads and probabilities of default.

5.1. Default probabilities and spreads

One model assumption is that creditors commit to lending for a small amount of time (see Section 3.5). Unfortunately, the model cannot be solved in closed-form under the more realistic assumption that governments issued long-term liabilities. In order to still benefit from the analytical framework of the previous sections, we approximate long-term spreads while solving for the price of a hypothetical contract with an arbitrary maturity that pays \$1 conditional upon no-default. We refer this contract

returns in excess of the risk-free rate as "spreads."[12]

Consider a time-horizon equal to $T-t$. During this term, the debt-to-GDP ratio, δ_t, may well overshoot the default boundary, $\bar{\delta}$. Let then $\pi(\delta, T-t)$ denote the probability of default at time-T when $\delta_t = \delta$. Default probabilities are probabilities of hitting times, i.e.,

$$\pi(\delta, T-t) = 1 - \Pr\left(\max_{u \in [t,T]} \delta_u \leq \bar{\delta} \middle| \delta_t = \delta \right),$$

where, by Proposition I (see Section 3), the debt-to-GDP ratio satisfies

$$d\delta_t = -\left(v(\delta_t) + \kappa - \iota(\delta_t)\right)\delta_t dt - \sigma \delta_t dW_t, \quad v(\delta_t) = s^1 \mathbf{1}_{\delta_t < \hat{\delta}} + s^2 \mathbf{1}_{\delta_t > \hat{\delta}}, \quad \delta_t < \bar{\delta}. \tag{15}$$

Spreads do not affect the government cost of capital, $r(\delta_t) \equiv i + \iota(\delta_t)$, where $\iota(\delta_t)$ is a fixed short-term spread determined at time-t (see Section 3.5.1). However, we investigate the asset pricing implications of a state-dependent short-term rate, $r(\delta_t)$. Let $1-L$ denote the recovery value of the bond upon default, where $L \in (0,1)$ is a positive constant. We assume that the recovery value, if any, is only paid off at T. Therefore, the bond price when the spread is θ, satisfies

$$e^{-(i+\theta)(T-t)} = E\left[e^{-\int_t^T r(\delta_u)du} \left(1 - L\left(1 - \mathbb{I}_{\max_{u \in [t,T]} \delta_u \leq \bar{\delta}}\right)\right) \middle| \delta_t = \delta \right]. \tag{16}$$

For example, assuming that $\iota(\cdot)$ is constant and equal to some $\hat{\iota}$, Eq. (16) leads to the following version of the "credit triangle" relation $1 - e^{-(\theta - \hat{\iota})(T-t)} = \pi(\delta, T-t) L$. That is, default probabilities and premiums are equivalent notions. However, these notions do not coincide in more general contexts: Section 5.6 considers a model in which the short-term spread depends on δ_t; in this model, the effects of discounting compound with those of default. We call "spread curve" the solution to (16) that maps debt to spreads, i.e., $\delta \longmapsto \theta(\delta, T-t, L)$.

5.2. Debt intolerance around fiscal tipping points

A striking property of the model is that the spread curve rises sharply around the fiscal tipping point. Figure 6 compares drifts and spreads predicted by the model with those predicted by an hypothetical "reduced-form" model with the same absorbing barrier $\bar{\delta}$: probabilities of hitting times do indeed increase disproportionately after some critical point. The model prediction may appear puzzling, as one might argue that a government in austerity is credibly signaling upcoming improved statistics. However, in the model, governments enter in austerity precisely *when* the probability of default is unacceptably high from their point of view. In other words, austerity occurs "too late" in that, by definition, it is the very same governments arguing for it being a time for austerity: governments

[12] This contract, then, is similar to a Credit Default Swap, in that it protects the buyer from the risk of default whilst not requiring the holder to be directly exposed to the underlying debt.

calibrate their tipping point $\hat{\delta}$ in order to spend as much time as possible in deficit.

There is a parallel between these properties and an old notion of "debt intolerance" discussed by Reinhart, Rogoff and Savastano (2003). These authors discuss contexts in which markets are "allergic" to lending money to Emerging Countries even in the presence of small debt-to-GDP ratios: a comparable "intake" of Advanced Economies debt might be acceptable. Similarly, in our model, the sudden surge in the spreads around $\hat{\delta}$ occurs when governments are in austerity. The point is that austerity is arriving late, i.e., precisely when default probabilities are already so high to have led governments to determine that it is not in their interest to keep on affording a deficit regime. Spreads are, then, much higher than they would have been if the government had run some other (from their perspective, suboptimal) moderate deficit policy.[13]

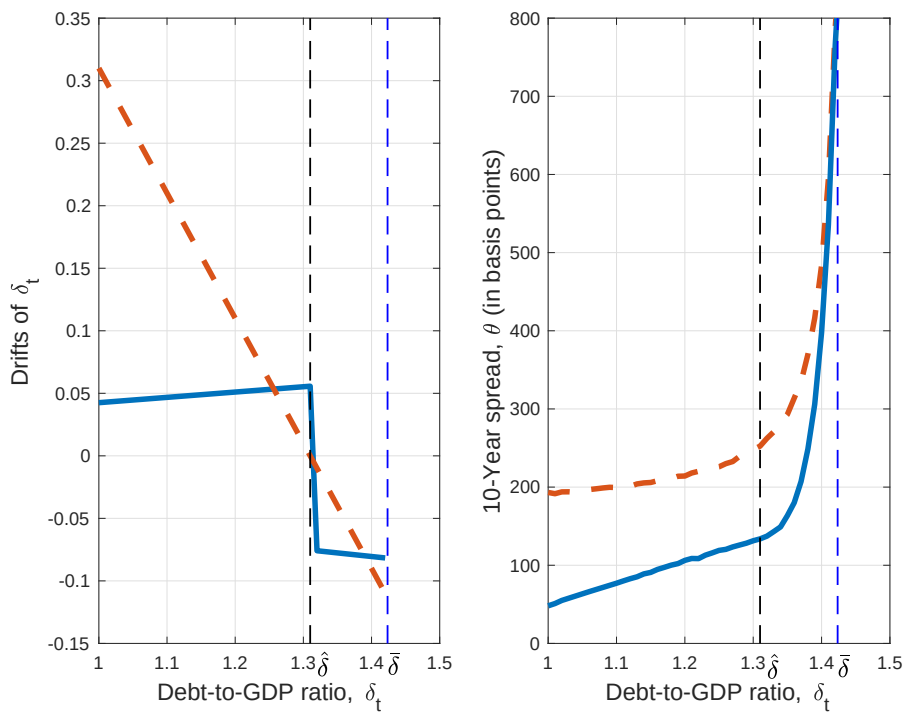

FIGURE 6: DEBT DRIFTS AND SPREADS AROUND FISCAL TIPPING POINTS. The left panel depicts the drift of δ_t predicted by the model (see Eq. (15)) (solid line) and by a reduced-form model with a linear drift equal to $\kappa_\delta(\hat{\delta} - \delta_t)$, and the same volatility and absorbing barrier $\bar{\delta}$ in (15), and where $\hat{\delta}$ denotes the fiscal tipping point (dashed). The right panel depicts the spreads predicted by the model (solid line) and the reduced-form model (dashed). Loss-given-default is equal to 60%. Parameter values are as in the legend of Figure 1 and, for the reduced-form model, $\kappa_\delta = 1$.

[13]To illustrate, the spreads resulting for $s^1 = -0.01$ would be just a few basis points for levels of debt less than 200% and remaining parameter values as in this section.

5.3. *Liquidity support and moral hazard*

Figure 7 depicts the spread as a function of δ_t for two default boundaries. The upper curve corresponds to an economy where liquidity conditions are tighter (i.e., a lower ℓ). It is an intuitive property: default probabilities decrease in markets that receive higher liquidity support, as in the case of a QE program. Assuming $\ell = 0.10$, a country with a debt-to-GDP ratio around 140% is on the verge of defaulting; when $\ell = 0.13$, 10-year spreads become less than 50 bps. This simple point may help explain the debt experience of the so-called "peripheral countries" during the QE programs of the 2010s, or that of countries such as Japan, where public debt is strongly supported by central banks and domestic investors. Note that the effects of better liquidity conditions on spreads are well visible even while δ_t is quite away from the default boundary.

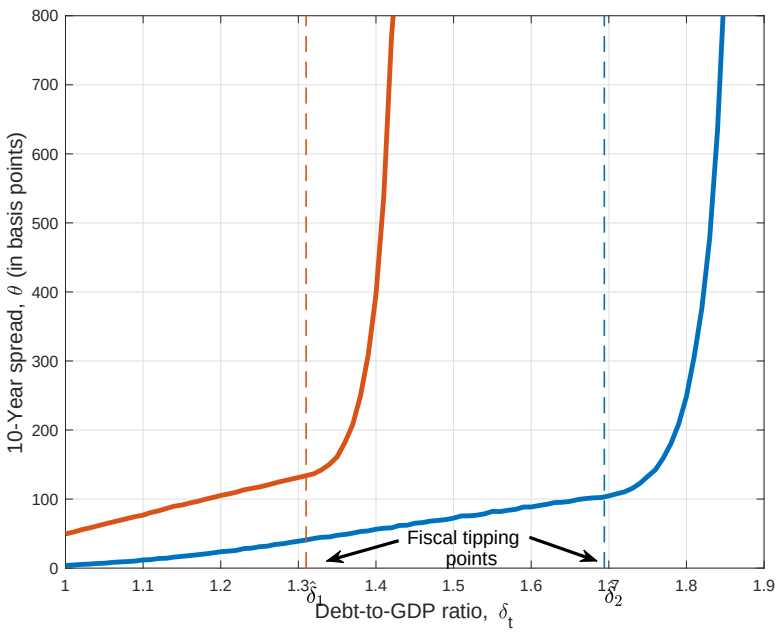

FIGURE 7: SPREADS AND LIQUIDITY SUPPORT. This picture depicts the spreads required to invest in a hypothetical 10-year zero coupon bond, against δ_t. Loss-given-default is equal to 60%. The uppermost curve obtains with liquidity parameter $\ell = 0.10$, and the lowermost with $\ell = 0.13$. Remaining parameter values are as in the legend of Figure 1. The vertical lines identify fiscal tipping points in these two cases ($\hat{\delta}_1$ when $\ell = 0.10$ and $\hat{\delta}_2$ when $\ell = 0.13$).

While spreads improve with ℓ, governments push back the fiscal tipping points (in this example, from $\hat{\delta}_1$ to $\hat{\delta}_2$) due to more elevated default boundaries: a higher liquidity support now creates moral hazard, providing governments with incentives to run deficits for longer. These properties provide rationale to policy programs such as the Ecb Outright Monetary Transactions, formulated

on September 6, 2012: this program conditioned debt support to governments engaging into fiscal adjustments. In terms of our model, spreads improve with a better liquidity support, although this improvement may be temporary, as governments would just enter a new season of deficits before they reach a new tipping point later. Section 5.6 revisits these problems when borrowing costs are state-dependent.

5.4. Volatility paradox

We study how macroeconomic volatility affects credit spreads. It is often argued that periods with low macroeconomic volatility (e.g., the Great Moderation) may contribute to markets with low risk premiums (see, e.g., Lettau, Ludvigson, and Wachter, 2008). However, it has also been explained that in good times, when volatility is low, endogenous risk and amplification mechanisms may build up (see Brunnermeier and Sannikov, 2014; and empirical evidence of Daníelsson, Valenzuela, and Zer, 2018, 2020). As a result, endogenous risk and expected returns may be positive even if exogenous volatility is small, a phenomenon dubbed "volatility paradox."

Our model predicts that governments push back their fiscal tipping points in economies with lower macroeconomic volatility (see Figure 3 in Section 4). Thus, risk is not uniquely determined by the fundamentals, but also by the government response to this risk: a feedback effect. Added to this property is a mechanical effect by which, with a less volatile output, δ_t is less likely to hit the default boundary following adverse exogenous shocks. Figure 8 shows that this mechanical effect dominates, such that spreads are increasing in volatility, consistent with the available empirical evidence (see Hilscher and Nosbusch, 2010). Figure 8 provides additional details regarding the magnitude of the feedback effect. Suppose volatility decreases; the left panel shows that the spread curve shifts down, although this shift could be much larger (up to the dashed line) if the government did not push back the fiscal tipping points. Likewise, the right panel shows that the spread curve increases with volatility, but that this increase could have been much larger (see the dashed line) if the governments did not act prudently by planning austerity earlier.

Figure 9 focusses on the relation between volatility and spreads. The upper curve is the model prediction. The lower curve is obtained by disregarding the government's feedback (i.e., by calculating the spread for each value of σ while assuming government determined the fiscal tipping point as if $\sigma = 0.05$). Without governments' feedback, spreads converge very quickly to zero as σ lowers. It is not the case with our model: as σ lowers, governments become prone to "take on more risks" by running deficits for longer, and spreads remain strictly positive (see the upper curve) even when σ is very small. This property helps rationalize the low and yet positive spreads on economies seemingly very unlikely to default, such as the U.S.. For example, Chernov, Schmid and Schneider (2020) develop a macro-based model that accounts for the spreads observed in the U.S. Our model provides a complementary view: economies with lower volatility provide governments with incentives to run more deficits.

This model prediction parallels the previous volatility paradox: governments run deficits longer when macroeconomic volatility is low, which contributes to positive default probabilities. This property also parallels others in the corporate finance literature. Notably, Acharya, Davydenko and Strebulaev (2012) find that riskier firms hold more cash, such that the correlation between cash and spreads is positive. In our model, debt commands a higher premium in riskier economies; however, governments run deficits for a shorter amount of time in these economies. Thus, our model predicts that debt carries higher spreads in economies where governments enter in austerity earlier. That is, running surpluses longer in our context parallels the property that a firm increases liquidity buffers in a riskier environment.

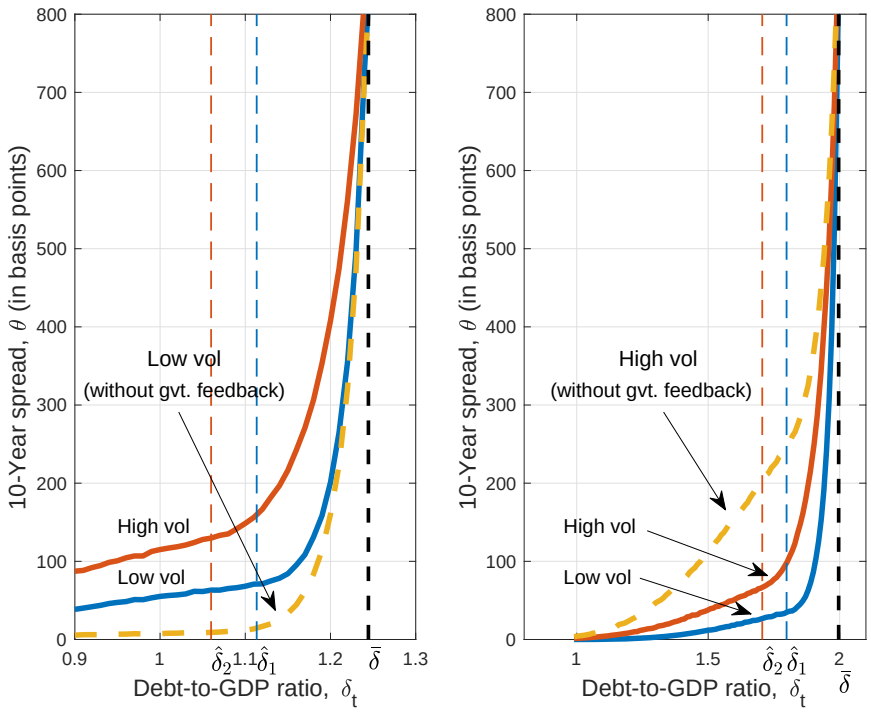

Figure 8: Spreads and macroeconomic volatility, I. This picture depicts the spreads required to invest in a hypothetical 10-year zero coupon bond, against δ_t. Loss-given-default is equal to 60%. *Left panel.* The "high vol" curve correspons to a GDP volatility $\sigma = 0.07$ and "low vol" corresponds to a decreased volatility $\sigma = 0.05$. The bottom, dashed curve depicts the spreads resulting from the assumption that the government does not respond to a lower volatility. Remaining parameters are as in the legend of Figure 1. *Right panel.* The curves depicts the spreads in a comparable comparative statics exercise, with $s^1 = -0.02$ and $s^2 = 0.06$.

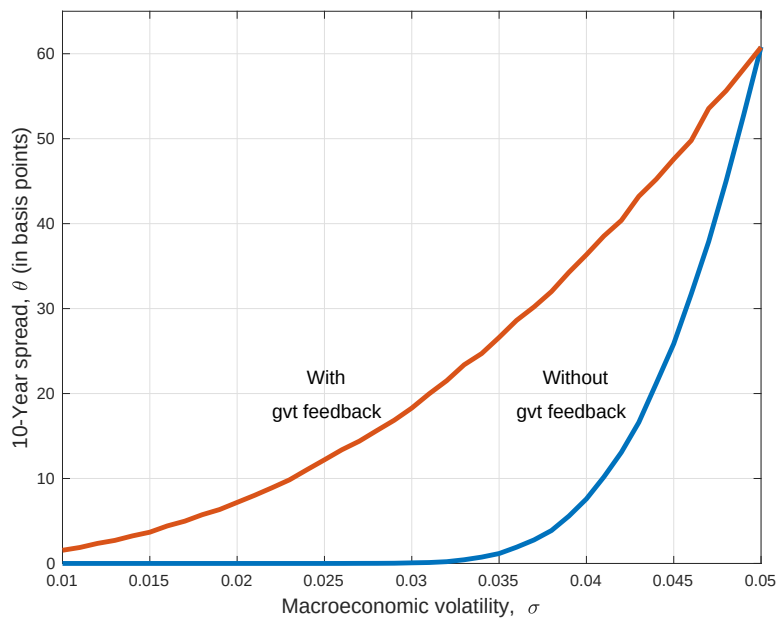

FIGURE 9: SPREADS AND MACROECONOMIC VOLATILITY, II. This picture depicts the spreads required to invest in a hypothetical 10-year zero coupon bond, against macroeconomic volatility. Loss-given-default is equal to 60%. The top curve obtains with the same parameter values in the legend of Figure 1, and fiscal tipping points are calculated with each value of σ in the horizontal axis. The bottom curve obtains when fiscal tipping points are determined while maintaining $\sigma = 0.05$. In both cases, debt-to-GDP fixed at $\delta_t = 1$.

5.5. Reforms

We study the impact of budget sizes on spreads while assuming a distortionary policy (see Section 3.7). We consider an economy with a negative growth, $\mu = -0.5\%$. The government then undertakes a series of costly initiatives that are assumed to increase growth. We term such policies "reforms." While empirical evidence suggests that fiscal policies might not have a permanent effect on output growth (see, e.g., Ramey, 2016), some literature has pointed to circumstances in which they could; DeLong and Summers (2012), for example, have reconsidered the role of governments in reversing hysteretic effects in economies trapped at a zero lower bound (see, also, Blanchard, 2018). Naturally, this section does not contribute to this debate. We simply explore how such reforms, if successful, would affect the cost of government borrowing. We consider two comparative statics experiments, resulting when a change in the budget size has, or has not, immediate effects.

5.5.1. *Unanticipated reforms*

Figure 10 depicts results when a reform is announced and immediately implemented, with a budget deficit increasing from $-s^1 = 0.05$ to 0.06, and the surplus immediately reducing from $s^2 = 0.05$ to 0.04. We assume that the reform is distortionary, in that it entails an immediate and permanent increase μ to 1.5%; we also assume that the effects of this reform operate immediately. Table 2 provides parameter values used in this example as well as in the benchmark.

	μ	s^1	s^2
Benchmark	-0.005	-0.05	0.05
Reform	0.015	-0.06	0.04
Reform (neutral)	-0.005	-0.06	0.04

Table 2: Assumptions regarding an unanticipated fiscal reform. This table provides parameter values used while implementing the fiscal reform experiments depicted in Figure 10.

The solid, blue line is the spread curve before the reform is announced and implemented. The solid, red line is the spread curve resulting from the reform. Note that, up to the fiscal tipping point ($\delta_t < \hat{\delta}$), the spread curve is substantially below that before the reform: the probability of default increases with a higher deficit (because the default boundary $\bar{\delta}$ gets tighter), but decreases due to higher growth, and the second effect dominates. However, as debt statistics deteriorate beyond the fiscal tipping point, the spread curve increases dramatically more than under the benchmark. The obvious reason is that, after the tipping point, when δ_t gets closer to $\bar{\delta}$, the risk of default becomes more and more relevant after the occurrence of negative shocks. This increased risk dwarfs the effects caused by an increased μ. Finally, the dashed, red line depicts the spread curve arising when the fiscal reform is growth-neutral: its only effect is to raise the probability of default.[14]

These properties parallel others known in the Laffer curve literature, which analyzes the extent to which lower taxation leads to higher growth and, sometimes, higher tax receipts and improved debt sustainability (see, e.g., D'Erasmo, Mendoza and Zhang, 2016). Our model provides novel predictions on default probabilities: a wider, but expansionary, budget may not affect spreads, provided debt is sufficiently low. However, it is an open question whether the distortionary effects in Figure 10 may be quantitatively matched by a production economy. In standard models, fiscal policies, whilst distortionary, may only affect the level of GDP, without leaving permanent effects on output growth, which is, instead, the key assumption in this section. We now examine how these predictions change when a reform is announced to take place conditional on the state of the economy.

[14]This property is not trivial, as governments plan to implement austerity earlier under the neutral reform. However, the default boundary would also occur earlier in this model. The Internet Appendix B provides results by which spreads may lower even under a neutral reform in the model with NPDL of Section 3.5.2, provided the debt limit is large enough.

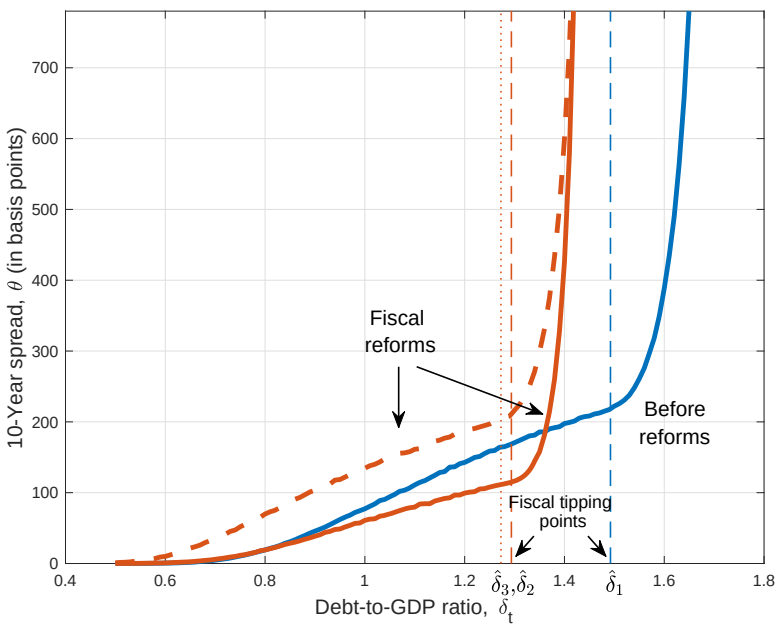

FIGURE 10: SPREADS AND FISCAL REFORMS, I. This picture depicts the spreads required to invest in a hypothetical 10-year zero coupon bond against δ_t. Loss-given-default is equal to 60%. The solid, blue curve is the spread curve before the reform, and obtains with parameter values equal to $s^{1/2} = \mp 0.05$, $i = 1\%$, $\mu = -0.005$, and remaining values as in the legend of Table 2. The solid and dashed, red curves depict the spread for the effective and the ineffective reform, and obtain with the same parameter values, except $s^1 = -0.06$ $s^2 = 0.04$ and $\mu = 0.015$ (solid) and $\mu = -0.005$ (dashed). The two red, vertical lines identify the fiscal tipping points under the effective ($\hat{\delta}_2$) and ineffective ($\hat{\delta}_3$) reform, and the blue, vertical line identifies the tipping point $\hat{\delta}_1$ before the reform.

5.5.2. Reform announcements

We study the effects of a reform announcement. Precisely, we assume that the government is in austerity ($\delta_t > \hat{\delta}$) and that the economy experiences a growth rate $\mu^2 \equiv -0.005$. The government, then, commits to implement a reform once debt statistics improve, only to undergo austerity again had debt statistics to overshoot the fiscal tipping point. Table 3 provides parameter values for this experiment. Because, now, the reform is *not* immediately implemented, we still have that $s^2 = 0.03$, as in the first line of Table 2. We assume that, once the reform is implemented (with deficit size $-s^1$ increasing by $|\Delta s| = 0.01$), growth will increase by $|\Delta\mu| \equiv 0.02$. As explained in Section 3.6, under conditions (see Appendix B), the model solution is as in Proposition I, but with parameter values equal to μ^1 in the deficit regime and μ^2 in the surplus regime: one condition requires a more limited range of variation for the budget sizes $[s^1, s^2]$ than that in Table 2. Figure 11 depicts our results.

	μ^1	μ^2	s^1	s^2
Benchmark	-0.005	-0.005	-0.03	0.03
Reform (announced)	0.015	-0.005	-0.04	0.03
Reform (unanticipated)	0.015	0.015	-0.04	0.02

TABLE 3: ASSUMPTIONS REGARDING ANNOUNCED AND UNANTICIPATED FISCAL REFORMS. This table provides parameter values used while implementing the fiscal reform experiments summarized in Figure 11. The parameters μ^1 and μ^2 are GDP growth rates in the deficit and the surplus regime.

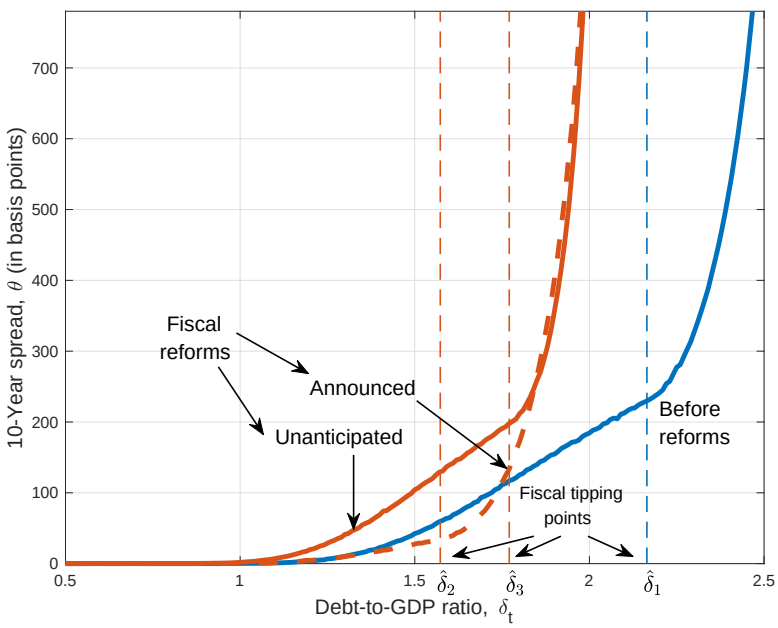

FIGURE 11: SPREADS AND FISCAL REFORMS, II. This picture depicts the spreads required to invest in a hypothetical 10-year zero coupon bond against δ_t. Loss-given-default is equal to 60%. The solid, blue curve identifies the spread curve before the reform, and obtains with parameter values equal to $s^{1/2} = \mp 0.03$, $i = 1\%$, $\mu = -0.005$, and remaining values as in the legend of Figure 1. The solid, red curve is the spread curve for the announced reform, and obtains with the same parameter values, except $\left(s^1, s^2\right) = (-0.04, 0.03)$, and growth $\mu^1 = 0.015$ (deficit regime) and $\mu^2 = -0.005$ (surplus). The solid, dashed curve is the spread curve for the unanticipated reform, and obtains with the same parameter values, except $\left(s^1, s^2\right) = (-0.04, 0.02)$, and $\mu = 0.015$. The two red, vertical lines identify the fiscal tipping points under the announced ($\hat{\delta}_2$) and unanticipated ($\hat{\delta}_3$) reforms, and the blue, vertical line identifies the tipping point $\hat{\delta}_1$ before the reform.

Note, now, that spreads are higher under the unanticipated reform than under the benchmark: while the fiscal tipping point lowers (from $\hat{\delta}_1$ to $\hat{\delta}_3$), growth is not high enough to compensate for the increased risk of default that is induced by a larger budget size, $-s^1$. However, spreads are lower under the announced reform than under the unanticipated: in the announced reform, austerity is more severe (i.e., s^2 is now higher) and growth is weaker, and the fiscal tipping point is lower, whence the lower spreads. In other words, governments act prudently knowing that the effects of a reform may only take place once debt statistics improve. In this particular example, spreads decrease compared to the benchmark, provided debt statistics are sufficiently low.

5.6. State-dependent interest rates and feedback loops

This section relaxes the assumption that interest rates are state-independent. For convenience, we assume that the short-term rate is $r(\delta_t) = i_1 + \iota(\delta_t) \equiv i_1 \mathbb{I}_{\delta_t < \hat{\delta}} + i_2 \mathbb{I}_{\delta_t \geq \hat{\delta}}$ ($i_1 = i = 1\%$ and $i_2 = 2.5\%$): the debt demand schedule becomes tighter as debt statistics deteriorate beyond the fiscal tipping point. This assumption is analytically convenient and its meaning is intuitive: interest rates rise precisely when even governments find δ_t (and the probability of default) to be unacceptably high. Therefore, the government policy is distortionary: interest rates depend on the fiscal regime (deficit or surplus).

A tighter demand schedule for larger values of δ_t affects governments' behavior. Figure 12 plots probabilities of default and *long*-term spreads against δ_t for both this model and the benchmark. The tipping point is now *much* lower in the model with state-dependent rates than in the benchmark. Moreover, probabilities of default may be much lower than when interest rates are constant (left panel), despite the intuitive fact that the default boundary decreases compared to the benchmark (see Eq. (B.1) in Appendix B). Spreads are larger, however (right panel). We now explain these properties.

In the model, a tighter demand for debt in the surplus region leads to a feedback property: once δ_t is in the surplus region, interest rates increase, which makes δ_t more likely to remain in the surplus region, thereby keeping interest rates high, over a vicious circle. In anticipation of these feedback effects, governments pull back from the deficit regime earlier than in the absence of feedbacks: in Figure 12, the fiscal tipping point lowers from $\hat{\delta}_1$ to $\hat{\delta}_2$, such that the implied distance to default $1 - C$ (see Section 4) increases, approximately, from 9% to 20%. The probabilities of default, then, decrease (compared to the benchmark case) while δ_t is sufficiently away from the default boundary. Note that spreads always increase in this model, reflecting heavier discounts (see Eq. (16)). However, the Internet Appendix B shows that in the presence of the same feedback effects of this section, spreads *do* decrease in the model with NPDL. The main reason is that, in this model, the default boundary is tighter in the presence of feedbacks; in the NPDL model, the default boundary is independent of the interest rate in the surplus region.

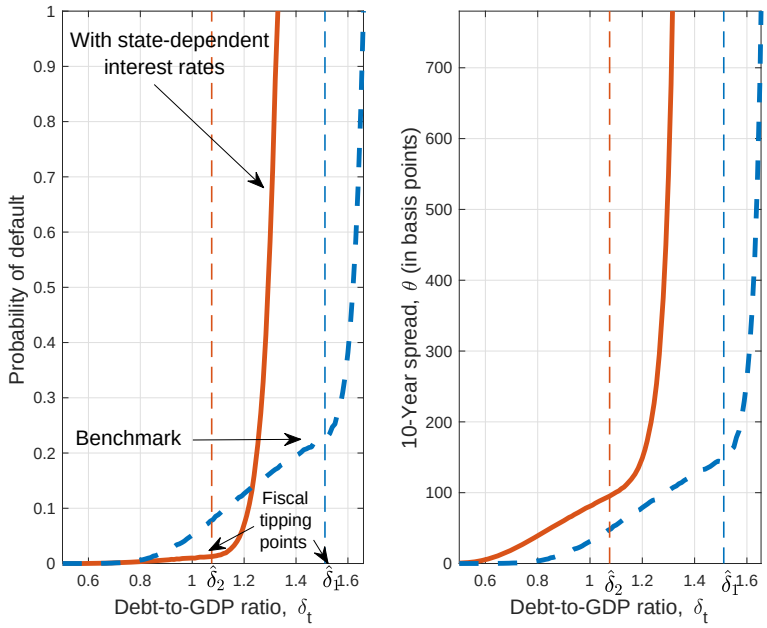

FIGURE 12: SPREADS AND STATE-DEPENDENT INTEREST RATES. The left panel depicts probabilities of default and the right panel depicts the spreads required to invest in a hypothetical 10-year zero coupon bond against the debt-to-GDP ratio, δ_t. Loss-given-default is equal to 60%. The red curve ("Benchmark") obtains with parameter values set equal to $s^{1/2} = \mp 0.03$, $i = 1\%$, $\mu = 1\%$, and other values as in the legend of Figure 1. The blue, dashed curve ("With state-dependent interest rates") obtains with the same parameter values, except that the interest rate changes across regimes, being equal to $i^1 = 1\%$ in the deficit regime and $i^2 = 2.5\%$ in the surplus regime. The two vertical lines identify the fiscal tipping points in the benchmark ($\hat{\delta}_1$) and in the state-dependent interest rate case ($\hat{\delta}_2$).

Blanchard (2019) explains that increasing debt involves small fiscal costs when interest rates are low. In our (benchmark) model, low interest rates, by incentivizing governments to run primary deficits for longer, lead δ_t to increase. The benchmark model can be thought of describing the case in which interest rates are stabilized by a program such as QE. In the model of this section, however, default probabilities would be much lower when δ_t is low: the higher borrowing costs in bad times act as a discipline device on governments' behavior. Naturally, as δ_t approaches the default boundary, these probabilities increase very quickly under market discipline; still, stabilizing interest rates would lead governments to accumulate more debt as a result of additional primary deficits, as illustrated by the benchmark curve in Figure 12. Thus, and similar to mechanisms analyzed in Section 5.3, the model rationalizes the claim that debt support may create moral hazard. For example, Eichengreen (2019) informally explains that QE policies, by artificially depressing the cost of government borrowing, may

lead governments to increase deficit spending more than under market discipline. Our model predicts that this is precisely the case.

6. Conclusion

Accumulation of national debt frequently makes the object of influential debates. For example, Reinhart and Rogoff (2010) conclusions are thought to have affected policymaking during the European debt crisis of the early 2010s. This paper deals with a related issue. When governments have preferences for deficits (or receive incentive-compatible mandates from citizens with these preferences), austerity plans may arrive too late to avert a debt crisis: governments implement austerity plans only when the probability of defaulting reaches a level that they, themselves, consider unacceptable. In particular, we find that fiscal tipping points occur at a distance-to-default between 10% and 20%. These figures are reminiscent of debates centered around the existence of ballpark estimates of debt sustainability. Indeed, our model predicts that fiscal tipping points depend on a variety of statistics that can be calibrated with actual data, such as economic growth, macroeconomic volatility, budget sizes, probabilities of re-entry after default, or governments' impatience; still, we find that the [10%,20%] range is robust to several alternative model calibrations.

Debt substainability is highly country-specific and there do not seem to exist general ways for assessing its degree. Our model provides a number of testable predictions regarding how fiscal tipping points, default probabilities and premiums vary with economic growth, macroeconomic volatility, probabilities of re-entry, the size of fiscal budgets, and fiscal reforms. For example, spreads increase with governments' short-sightedness, or debt market illiquidity. Governments also push back fiscal tipping points in economies with less macroeconomic volatility, such that spreads remain positive even when macroeconomic risk is very small, and are, then, higher in countries that implement austerity earlier—two public debt versions of the "volatility paradox." Liquidity support in bad times (i.e., when debt is high) may temporarily help governments maintain low spreads; however, they also create moral hazard, incentivizing governments to run deficits for longer, and increasing future debt and spreads accordingly. Exploring the empirical content of these theoretical propositions is left for future research.

Appendix A: Deficit policy and exogenous default

Proof of Proposition I. Consider the Bellman equation in Eq. (7),

$$0 = \frac{1}{2}\sigma^2\delta^2 V''(\delta) - (s(\delta) + \kappa)\,\delta V'(\delta) - \rho V(\delta) + s(\delta), \tag{A.1}$$

where

$$s(\delta) = \begin{cases} s^1, & \text{for all } \delta : V'(\delta)\,\delta < 1 \\ s^2, & \text{for all } \delta : V'(\delta)\,\delta > 1 \end{cases} \tag{A.2}$$

We search for a solution of this type and conjecture that this solution is such that the mapping $\delta \longmapsto V'(\delta)\,\delta$ is increasing and such that $V'(\delta)\,\delta < 1$ for arbitrarily small values of δ. These properties will be verified later. The solution to Eq. (A.1) is $V(\delta) = V_{\mathcal{D}}(\delta)\,\mathbf{1}_{\delta<\hat{\delta}} + V_{\mathcal{S}}(\delta)\,\mathbf{1}_{\delta>\hat{\delta}}$, where

$$V_{\mathcal{D}}(\delta) = \frac{s^1}{\rho} + A_{\mathcal{D}1}\delta^{m_{\mathcal{D}1}} + A_{\mathcal{D}2}\delta^{m_{\mathcal{D}2}}, \quad V_{\mathcal{S}}(\delta) = \frac{s^2}{\rho} + A_{\mathcal{S}1}\delta^{m_{\mathcal{S}1}} + A_{\mathcal{S}2}\delta^{m_{\mathcal{S}2}}, \tag{A.3}$$

for four constants $A_{\mathcal{D}1}$, $A_{\mathcal{D}2}$, $A_{\mathcal{S}1}$, $A_{\mathcal{S}2}$, and

$$m_{\mathcal{X}1/\mathcal{X}2} = \frac{1}{2} + \frac{s_{\mathcal{X}} + \kappa}{\sigma^2} \mp \sqrt{\left(\frac{1}{2} + \frac{s_{\mathcal{X}} + \kappa}{\sigma^2}\right)^2 + \frac{2\rho}{\sigma^2}}, \quad s_{\mathcal{D}} \equiv s^1, \quad s_{\mathcal{S}} \equiv s^2. \tag{A.4}$$

It is straightforward to show that $m_{\mathcal{D}1} < 0$, $m_{\mathcal{D}2} > 0$, $m_{\mathcal{S}1} < 0$, and that $m_{\mathcal{S}2} > 0$. Note that, by (2), $V(\delta) \geq \frac{s^1}{\rho}$ for all $\delta > 0$. Therefore, $A_{\mathcal{D}1} = 0$ and, below, we show that $A_{\mathcal{D}2} > 0$.

Next, we impose the following conditions, which all have to hold, provided governments default in the surplus regime.

- (i) Free boundary: $V'_{\mathcal{D}}(\delta)\,\delta|_{\delta=\hat{\delta}} = 1$;
- (ii) Value matching: $V_{\mathcal{D}}(\hat{\delta}) = V_{\mathcal{S}}(\hat{\delta})$;
- (iii) Smooth-pasting: $V'_{\mathcal{D}}(\delta)|_{\delta=\hat{\delta}} = V'_{\mathcal{S}}(\delta)|_{\delta=\hat{\delta}}$;
- (iv) Default boundary: $V(\bar{\delta}) = \bar{V}$.

We need to apply the default boundary (iv) in two distinct cases resulting according to whether default occurs in the deficit or in the surplus regime. We analyze these cases below. For brevity, this appendix relies on the assumption that $\bar{V} \leq \frac{s^2}{\rho}$; in Internet Appendix A, we show that the model solution is still the same when $\bar{V} > \frac{s^2}{\rho}$.

- If default occurs in the deficit regime, the utility costs are entirely determined by (iv), with $V(\bar{\delta}) = V_{\mathcal{D}}(\bar{\delta}) = \bar{V}$. Default occurs in the deficit regime provided $\hat{\delta} > \bar{\delta}$, where $\hat{\delta}$ satisfies (i). Because, as argued above, $A_{\mathcal{D}1} = 0$, then, the two conditions (i) and (iv) are

$$m_{\mathcal{D}2}A_{\mathcal{D}2}\hat{\delta}^{m_{\mathcal{D}2}} = 1, \quad \bar{V} = \frac{s^1}{\rho} + A_{\mathcal{D}2}\bar{\delta}^{m_{\mathcal{D}2}}, \tag{A.5}$$

  which leave the following solution for the tipping point $\hat{\delta}$ and the coefficient $A_{\mathcal{D}2}$,

$$\hat{\delta} = \left(m_{\mathcal{D}2}\left(\bar{V} - \frac{s^1}{\rho}\right)\right)^{-\frac{1}{m_{\mathcal{D}2}}}\bar{\delta}, \quad A_{\mathcal{D}2} = \left(\bar{V} - \frac{s^1}{\rho}\right)\bar{\delta}^{-m_{\mathcal{D}2}}. \tag{A.6}$$

Therefore, default occurs in the deficit regime, provided $\hat{\delta} > \bar{\delta}$, that is, when

$$m_{\mathcal{D}2}\left(\bar{V} - \frac{s^1}{\rho}\right) < 1. \tag{A.7}$$

We now verify that if, instead, (A.7) does not hold, default does indeed occur in the surplus regime. We verify the claim while analyzing the utility costs in the surplus regime.

- If default occurs in the surplus regime, conditions (i)-(iv) hold, with $V(\bar{\delta}) = V_{\mathcal{S}}(\bar{\delta}) = \bar{V}$ in (iv). Below, we shall establish that the thusly determined function also satisfies *(v)* $V''_{\mathcal{D}}(\delta)|_{\delta=\hat{\delta}} = V''_{\mathcal{S}}(\delta)|_{\delta=\hat{\delta}}$, a super-contact condition. The first four conditions are

$$\begin{cases} m_{\mathcal{D}2}A_{\mathcal{D}2}\hat{\delta}^{m_{\mathcal{D}2}} & = & 1 & (i) \\ \dfrac{s^1}{\rho} + A_{\mathcal{D}2}\hat{\delta}^{m_{\mathcal{D}2}} & = & \dfrac{s^2}{\rho} + A_{\mathcal{S}1}\hat{\delta}^{m_{\mathcal{S}1}} + A_{\mathcal{S}2}\hat{\delta}^{m_{\mathcal{S}2}} & (ii) \\ m_{\mathcal{D}2}A_{\mathcal{D}2}\hat{\delta}^{m_{\mathcal{D}2}} & = & m_{\mathcal{S}1}A_{\mathcal{S}1}\hat{\delta}^{m_{\mathcal{S}1}} + m_{\mathcal{S}2}A_{\mathcal{S}2}\hat{\delta}^{m_{\mathcal{S}2}} & (iii) \\ \bar{V} & = & \dfrac{s^2}{\rho} + A_{\mathcal{S}1}\bar{\delta}^{m_{\mathcal{S}1}} + A_{\mathcal{S}2}\bar{\delta}^{m_{\mathcal{S}2}} & (iv) \end{cases} \tag{A.8}$$

Note that, because $m_{\mathcal{D}2} > 0$, then, by (i), it must be the case that $A_{\mathcal{D}2} > 0$. Moreover, replacing (i) into (ii)-(iii) leaves a system of two equations with the two unknowns $y_i \equiv A_{\mathcal{S}i}\hat{\delta}^{m_{\mathcal{S}i}}$, $i = 1, 2$, solved by

$$Y_1 = \frac{Bm_{\mathcal{S}2} - 1}{m_{\mathcal{S}2} - m_{\mathcal{S}1}}, \quad Y_2 = \frac{1 - Bm_{\mathcal{S}1}}{m_{\mathcal{S}2} - m_{\mathcal{S}1}}, \quad B \equiv \frac{s^1 - s^2}{\rho} + \frac{1}{m_{\mathcal{D}2}}. \tag{A.9}$$

Replacing the solution for Y_1 into (iv) yields the following equation in $\hat{\delta}$

$$\bar{V} - \frac{s^2}{\rho} = \varphi(\hat{\delta}/\bar{\delta}), \quad \varphi(x) \equiv Y_1 x^{-m_{\mathcal{S}1}} + Y_2 x^{-m_{\mathcal{S}2}}. \tag{A.10}$$

Assume, now, that (A.7) does not hold. We claim that $Y_1 < 0$. Indeed, note that

$$1 < m_{\mathcal{D}2}\left(\bar{V} - \frac{s^1}{\rho}\right) \leq m_{\mathcal{D}2}\left(\frac{s^2 - s^1}{\rho}\right).$$

This inequality implies that $B < 0$ and, then, that $Y_1 < 0$. Moreover, by direct but lengthy calculations in the Internet Appendix A, we find that $Y_2 > 0$. Thus, the function $\varphi(x)$ is strictly decreasing. Moreover, we have that $\lim_{x \longrightarrow 0} \varphi(x) = \infty$ and $\lim_{x \longrightarrow \infty} \varphi(x) = -\infty$, such that there exists a solution x to (A.10) depending on $\bar{V}$, say $x(\bar{V})$, which is decreasing in $\bar{V}$. Moreover,

$$\varphi(x(\bar{V})) = \bar{V} - \frac{s^2}{\rho} > \frac{s^1 - s^2}{\rho} + \frac{1}{m_{\mathcal{D}2}} = B = Y_1 + Y_2 = \varphi(1).$$

Therefore, we have shown that if (A.7) does not hold, $x(\bar{V}) < 1$, or $\hat{\delta} < \bar{\delta}$, that is, default occurs in the surplus regime, as previously claimed. Let, then, $\hat{\delta}$ denote the solution to Eq. (A.10). The solutions to the remaining coefficients are

$$A_{\mathcal{D}2} = \frac{\hat{\delta}^{-m_{\mathcal{D}2}}}{m_{\mathcal{D}2}}, \quad A_{\mathcal{S}i} = Y_i\hat{\delta}^{-m_{\mathcal{S}i}}, \quad i = 1, 2. \tag{A.11}$$

Thus, Proposition I follows by Eq. (A.3), and the government utility costs $V(\delta)$ are as follows:

- If (A.7) holds, the government defaults in the deficit regime, and

$$V(\delta) = V_{\mathcal{D}}(\delta) = \frac{s^1}{\rho} + A_{\mathcal{D}2}\delta^{m_{\mathcal{D}2}}, \tag{A.12}$$

  where $A_{\mathcal{D}2}$ is as in (A.6);

- If (A.7) does not hold, the government defaults in the surplus regime, and

$$V(\delta) = V_{\mathcal{D}}(\delta)\,\mathbf{1}_{\delta<\hat{\delta}} + V_{\mathcal{S}}(\delta)\,\mathbf{1}_{\delta>\hat{\delta}},$$

  where $A_{\mathcal{D}2}$, $A_{\mathcal{S}1}$ and $A_{\mathcal{S}2}$ are as in (A.11), $V_{\mathcal{D}}(\delta)$ is as in (A.12), and the threshold value equals $\hat{\delta} = x\left(\bar{V}\right)\bar{\delta}$, where $x\left(\bar{V}\right)$ is the solution to (A.10).

Note that, in the special case in which the utility costs at default are $\bar{V} = \frac{s^2}{\rho}$, the threshold value $\hat{\delta}$ (resulting when default occurs in the surplus regime) can be solved in closed-form as

$$\hat{\delta} = \left(-\frac{Y_2}{Y_1}\right)^{\frac{1}{m_{\mathcal{S}2}-m_{\mathcal{S}1}}}\bar{\delta}.$$

It is easy to check that, when (A.7) (with $\bar{V} = \frac{s^2}{\rho}$) holds, then $B > 0$, such that $Y_2 > -Y_1$ or $\hat{\delta} > \bar{\delta}$: governments then default in the deficit regime. Likewise, when (A.7) (with $\bar{V} = \frac{s^2}{\rho}$) does not hold, then, $\hat{\delta} < \bar{\delta}$, and governments default in the surplus regime.

Finally, note that Eq. (A.10) is satisfied by the ratio $\hat{\delta}/\bar{\delta}$ also when $\bar{V} > \frac{s^2}{\rho}$. In the Internet Appendix A, we show that $Y_1 < 0$ and $Y_2 \geq 0$ in this case too, such that the function φ is still decreasing. Therefore, the ratio $\hat{\delta}/\bar{\delta}$ solution to Eq. (A.10) is decreasing with $\bar{V}$, which establishes the last statement of the proposition.

Super-contact condition. The functions $V_{\mathcal{D}}(\delta)$ and $V_{\mathcal{S}}(\delta)$ both satisfy the ordinary differential equation (A.1). Subtracting the equation for $V_{\mathcal{S}}(\delta)$ from the equation for $V_{\mathcal{D}}(\delta)$, and evaluating the result at $\delta = \hat{\delta}$, leaves

$$0 = \frac{1}{2}\sigma^2\hat{\delta}^2\left(V''_{\mathcal{D}}(\hat{\delta}) - V''_{\mathcal{S}}(\hat{\delta})\right) - \left(s^1 V'_{\mathcal{D}}(\hat{\delta}) - s^2 V'_{\mathcal{S}}(\hat{\delta})\right)\hat{\delta} - \kappa\hat{\delta}\left(V'_{\mathcal{D}}(\hat{\delta}) - V'_{\mathcal{S}}(\hat{\delta})\right) - \rho\left(V_{\mathcal{D}}(\hat{\delta}) - V_{\mathcal{S}}(\hat{\delta})\right) + \left(s^1 - s^2\right).$$

Using conditions (i) and (ii) in (A.8) leaves

$$0 = \frac{1}{2}\sigma^2\hat{\delta}^2\left(V''_{\mathcal{D}}(\hat{\delta}) - V''_{\mathcal{S}}(\hat{\delta})\right) - \left(s^1 - s^2\right) + \left(s^1 - s^2\right),$$

such that $V''_{\mathcal{D}}(\delta)|_{\delta=\hat{\delta}} = V''_{\mathcal{S}}(\delta)|_{\delta=\hat{\delta}}$.

Monotonicity of marginal utility costs. Define the marginal utility costs as $\mathcal{V}(\delta) \equiv V'(\delta)\,\delta$, where $V(\delta)$ is as in Proposition I, and $V_{\mathcal{D}}(\delta)$ and $V_{\mathcal{S}}(\delta)$ are as in (A.3). It is immediate to see that, for all $\delta \leq \hat{\delta}$, $\mathcal{V}(\delta) \leq 1$ and $\mathcal{V}'(\delta) > 0$. We claim that $\mathcal{V}'(\delta) > 0$ for $\delta > \hat{\delta}$ too, which would also establish that $\mathcal{V}(\delta) > 1$ for all $\delta > \hat{\delta}$. Indeed, note that

$$0 < \frac{d}{d\delta}\left(V'_{\mathcal{D}}(\delta)\,\delta\right)\Big|_{\delta=\hat{\delta}} = \mathcal{V}'(\delta)|_{\delta=\hat{\delta}} = \left(V''_{\mathcal{S}}(\delta)\,\delta\right)|_{\delta=\hat{\delta}} + V'_{\mathcal{S}}(\delta)|_{\delta=\hat{\delta}} \equiv \Phi(\hat{\delta}),$$

where the first inequality follows by a simple calculation, and the equalities follow by condition (iii) and because V satisfies the super-contact condition. Thus, $\Phi(\hat{\delta}) > 0$, where, using the expression for $V_{\mathcal{S}}$, $\Phi(\delta) \equiv$

$m_{\mathcal{S}1}^2 A_{\mathcal{S}1}\delta^{m_{\mathcal{S}1}-1} + m_{\mathcal{S}2}^2 A_{\mathcal{S}2}\delta^{m_{\mathcal{S}2}-1}$; that is, using the expressions of $A_{\mathcal{S}i}$ in (A.11),

$$1 > Q \equiv \left(\frac{m_{\mathcal{S}1}}{m_{\mathcal{S}2}}\right)^2 \frac{-Y_1}{Y_2} > 0.$$

Next, we show that $\Phi(\delta) > 0$ for all $\delta > \hat{\delta}$, thereby establishing our claim. For all $\delta > \hat{\delta}$, we have that $\Phi(\delta) > 0$ if and only if

$$\delta^{m_{\mathcal{S}2}-m_{\mathcal{S}1}} > -\left(\frac{m_{\mathcal{S}1}}{m_{\mathcal{S}2}}\right)^2 \frac{A_{\mathcal{S}1}}{A_{\mathcal{S}2}} = Q\hat{\delta}^{m_{\mathcal{S}2}-m_{\mathcal{S}1}}.$$

This inequality does indeed hold for all $\delta > \hat{\delta}$, because $m_{\mathcal{S}2} > 0$, $m_{\mathcal{S}1} < 0$ and $Q < 1$. Therefore, marginal utility costs are increasing for all $\delta \leq \bar{\delta}$.

Proof of Eq. (10). Consider, first, the following heuristic arguments. Assume that time increments are discrete and equal to Δt. At each point in time during the exclusion period, $t \geq \tau$, the government utility costs are given by the imminent costs, $\epsilon\Delta t$, plus the utility costs expected for the next period, viz

$$\mathcal{C}_d(\bar{\delta}) = \epsilon\Delta t + e^{-\rho\Delta t}[(1 - e^{-\vartheta\Delta t})V(\gamma\bar{\delta}) + e^{-\vartheta\Delta t}\mathcal{C}_d(\bar{\delta})]. \tag{A.13}$$

The second term on the R.H.S. summarizes the expected discounted utility costs: with probability $1 - e^{-\vartheta\Delta t}$, the government will re-gain access to markets, in which case its utility costs equal $V(\gamma\bar{\delta})$; with probability $e^{-\vartheta\Delta t}$, the government will be stuck in exclusion, incurring in utility costs still equal to $\mathcal{C}_d(\bar{\delta})$. Eq. (10) follows, heuristically, by taking limits in (A.13) and by re-arranging terms.

A rigorous proof of Eq. (10) is as follows. At each instant of time t, the probability that the government is still in a state of financial exclusion is $e^{-\vartheta t}$, whereas the probability of re-entry is $\vartheta e^{-\vartheta t}$. Therefore, the utility costs at default are

$$\mathcal{C}(\delta_\tau) = \xi\delta_\tau + \int_0^\infty e^{-\rho t}\left[e^{-\vartheta t}\epsilon + \vartheta e^{-\vartheta t}V(\gamma\delta_\tau)\right] dt, \tag{A.14}$$

where the terms inside the brackets in the integral are the expected instantaneous utility cost at time $t > \tau$. Eq. (10) follows by Eq. (A.14).

Solution with re-entry. We provide an explicit characterization of the solution in Proposition I, based on the expression for $\bar{V}$ in Eq. (10). Deficit may occur in the default or surplus regime, according to whether (A.7) holds.

- *Default in the deficit regime.* By replacing the first into the second equality of (A.5),

$$\frac{s^1}{\rho} + \frac{1}{m_{\mathcal{D}2}}x^{-m_{\mathcal{D}2}} = \bar{V} = \frac{\epsilon}{\rho+\vartheta} + \frac{\vartheta}{\rho+\vartheta}V(\gamma\bar{\delta}) + \xi\bar{\delta}, \quad V(\gamma\bar{\delta}) = \frac{s^1}{\rho} + \frac{\gamma^{m_{\mathcal{D}2}}}{m_{\mathcal{D}2}}x^{-m_{\mathcal{D}2}},$$

  where we have defined $x \equiv \frac{\hat{\delta}}{\bar{\delta}}$, used the expression for the utility costs at default in Eq. (10), and the assumption that re-entry occurs in the deficit regime. That is,

$$\xi\bar{\delta} + \frac{\epsilon - s^1}{\rho+\vartheta} = \frac{\rho + \vartheta(1-\gamma^{m_{\mathcal{D}2}})}{m_{\mathcal{D}2}(\rho+\vartheta)}x^{-m_{\mathcal{D}2}}. \tag{A.15}$$

  It is immediate to show that, given the assumptions on the sign of all coefficients, and $m_{\mathcal{D}2} > 0$, there exists a unique solution to Eq. (A.15), denoted as $x_{\mathcal{D}}$, and the Internet Appendix A confirms that $x_{\mathcal{D}} > 1$. Thus, $\hat{\delta}_{\mathcal{D}} = x_{\mathcal{D}}\bar{\delta}$, and

$$A_{\mathcal{D}2} = \frac{\hat{\delta}_{\mathcal{D}}^{-m_{\mathcal{D}2}}}{m_{\mathcal{D}2}}. \tag{A.16}$$

Eq. (A.16) completes the solution for the government utility costs in the case default occurs in the deficit regime.

- *Default in the surplus regime.* Replacing the expression of $\bar{V}$ in Eq. (10) into Eq. (A.10) leaves

$$\frac{\epsilon}{\rho+\vartheta}+\frac{\vartheta}{\rho+\vartheta}\left(\frac{s^1}{\rho}+\frac{\gamma^{m_{\mathcal{D}2}}}{m_{\mathcal{D}2}}x^{-m_{\mathcal{D}2}}\right)+\xi\bar{\delta}-\frac{s^2}{\rho}=\varphi(x), \tag{A.17}$$

where the function φ is as in (A.10) and, again, $x\equiv\frac{\hat{\delta}}{\bar{\delta}}$. By the proof of Proposition I, it follows that there exists a unique solution $x_{\mathcal{S}}$ to Eq. (A.17), such that $\hat{\delta}_{\mathcal{S}}=x_{\mathcal{S}}\bar{\delta}$, and

$$A_{\mathcal{D}2}=\frac{\hat{\delta}_{\mathcal{S}}^{-m_{\mathcal{D}2}}}{m_{\mathcal{D}2}},\quad A_{\mathcal{S}i}=Y_i\hat{\delta}_{\mathcal{S}}^{-m_{\mathcal{S}i}},\quad i=1,2, \tag{A.18}$$

where Y_i are as in (A.9). Eq (A.18) completes the solution for the government utility costs in the case of default in the surplus regime.

Incentive-compatible mandates. We provide proofs for Section 3.2.2. The following lemma provides the dynamics of the governments' continuation utility and a condition for incentive-compatibility of their mandate. We have:

Lemma A.1. *There exists some measurable process ϕ_t such that Eq. (4) in the main text holds, and the governments' mandate is incentive-compatible if and only if $\phi_t\geq\lambda$ for all t.*

Proof. Note that

$$\bar{q}_t\equiv E_t\left[\int_0^\tau e^{-\chi u}\frac{d\hat{G}_u}{D_u}+e^{-\chi(\tau-t)}R\right]=\int_0^t e^{-\chi u}\frac{d\hat{G}_u}{D_u}+e^{-\chi t}q_t. \tag{A.19}$$

Because $\bar{q}_t$ is a martingale, then, by the Martingale Representation Theorem, there exists a measurable process ϕ_t such that

$$\bar{q}_t=\bar{q}_0+\psi\sigma\int_0^t e^{-\chi u}\phi_u dW_u. \tag{A.20}$$

Therefore, by expanding (A.19) and (A.20), and equating the results, yields (4). Next, define temporarily $\hat{q}_t$ as the government continuation utility when the cumulative surplus history is $(\hat{\mathcal{S}}_u)_{u\leq t}$. We have

$$\begin{aligned}
d\hat{q}_t&=\chi\hat{q}_t dt-\frac{d\hat{G}_t}{D_t}+\psi\sigma\phi_t dW_t\\
&=\chi\hat{q}_t dt-\left(\frac{dG_t}{D_t}+\frac{\lambda}{D_t}(d\mathcal{S}_t-d\hat{\mathcal{S}}_t)\right)+\frac{\phi_t}{D_t}\left(d\mathcal{S}_t-S_t dt\right)\\
&=\chi\hat{q}_t dt-\frac{dG_t}{D_t}+\frac{1}{D_t}\left(\phi_t-\lambda\right)\left(d\mathcal{S}_t-d\hat{\mathcal{S}}_t\right)+\frac{\phi_t}{D_t}\left(d\hat{\mathcal{S}}_t-S_t dt\right).
\end{aligned}$$

By integrating,

$$\hat{q}_0=q_0+E\left[\int_0^\tau e^{-\chi t}\frac{1}{D_t}\left(\lambda-\phi_t\right)\left(d\mathcal{S}_t-d\hat{\mathcal{S}}_t\right)\right]-E\left[\int_0^\tau e^{-\chi t}\frac{\phi_t}{D_t}\left(d\hat{\mathcal{S}}_t-S_t dt\right)\right].$$

Therefore, the governments' mandate is incentive-compatible if and only if $\phi_t\geq\lambda$, in which case $d\mathcal{S}_t=d\hat{\mathcal{S}}_t$

and, then, $\hat{q}_0 = q_0$. ■

Now, based on Lemma A.1, we can re-write the program in (5) in the main text as

$$\mathcal{V}(q,\delta) = \inf_{\phi_t \geq \lambda, dG_t \geq 0, s_t \in [s^1, s^2]} E\left[\int_0^\infty e^{-\rho t}\left(s_t dt + dG_{Dt}\right)\right],$$

subject to

$$\begin{cases} dq_t &= \chi q_t dt - dG_{Dt} + \psi\sigma\phi_t dW_t \\ d\delta_t &= -\left(s_t + \kappa\right)\delta_t dt - \sigma\left(1+\psi\right)\delta_t dW_t \end{cases}$$

where $dG_{Dt} \equiv \frac{dG_t}{D_t}$. The Bellman's equation for this problem is

$$0 = \inf_{\phi \geq \lambda, dG \geq 0, s \in [s^1, s^2]} \left[\left(L\mathcal{V}(q,\delta) + s - \rho\mathcal{V}(q,\delta)\right)dt + \left(1 - \mathcal{V}_q(q,\delta)\right)dG_D\right], \quad \mathcal{V}(0,\delta) = L.$$

where

$$L\mathcal{V}(q,\delta) = \begin{pmatrix} \frac{1}{2}\psi^2\sigma^2\phi^2\mathcal{V}_{qq}(q,\delta) + \chi q\mathcal{V}_q(q,\delta) \\ +\frac{1}{2}\sigma^2\left(1+\psi\right)^2\delta^2\mathcal{V}_{\delta\delta}(q,\delta) - \left(s+\kappa\right)\delta\mathcal{V}_\delta(q,\delta) \\ -\sigma^2\psi\left(1+\psi\right)\phi\delta\mathcal{V}_{q\delta}(q,\delta) \end{pmatrix},$$

and subscripts denote partial derivatives. It is easy to see that $\mathcal{V}(q,\delta)$ is as in Eq. (6), such that $V(\delta)$ is as in (2), and $\mathcal{G}(q)$ solves the Hamilton-Jacobi-Bellman variational inequality

$$0 = \inf_{\phi \geq \lambda}\left\{\frac{1}{2}\psi^2\sigma^2\phi^2\mathcal{G}''(q) + \chi q\mathcal{G}'(q) - \rho\mathcal{G}(q), \mathcal{G}'(q) - 1\right\}, \quad \mathcal{G}(0) = L. \tag{A.21}$$

The policy is to implement a cumulative compensation $\ell(q^o, t)$, the local time of q_t at q^o, where q^o is such that $\mathcal{G}'(q^o) = 1$ (free boundary) and $\mathcal{G}''(q^o) = 0$ (super-contact).

Lemma A.2. *The problem in (A.21) is solved by $\phi = \lambda$.*

Proof. It is sufficient to show that $\mathcal{G}(q)$ is convex. The proof extends that in Mele (2022, Chapter 5). Indeed, assume that $\mathcal{G}'(q) < 1$, and differentiate the differential equation in (A.21) and evaluate the result at $q = q^o$, obtaining

$$0 = \frac{1}{2}\psi^2\sigma^2\phi^2\mathcal{G}'''(q^o) + \chi\mathcal{G}'(q^o) + \chi q\mathcal{G}''(q^o) - \rho\mathcal{G}'(q^o) = \frac{1}{2}\psi^2\sigma^2\phi^2\mathcal{G}'''(q^o) + \chi - \rho.$$

Then, because citizens are more patient than the government, $\mathcal{G}'''(q^o) < 0$. This inequality implies that $\mathcal{G}''(q^o - \Delta q^o) > \mathcal{G}''(q^o) = 0$, for small $\Delta q^o > 0$. Next, we show that $\mathcal{G}''(q) > 0$ for all $q < q^o$. Indeed, suppose the contrary, i.e., that there exists a $q_* : \mathcal{G}''(q) > 0$ for all $q \in (q_*, q^o)$ and $\mathcal{G}''(q_*) = 0$. These assumptions imply that $\mathcal{G}'''(q_*) > 0$. Now, differentiating the differential equation in (A.21), evaluating the result at $q = q_*$ and using $\mathcal{G}''(q_*) = 0$,

$$\mathcal{G}'(q_*) = \frac{1}{2}\psi^2\sigma^2\phi^2\frac{\mathcal{G}'''(q_*)}{\rho - \chi} < 0. \tag{A.22}$$

Finally, evaluating the differential equation in (A.21) at $q = q_*$, and using $\mathcal{G}''(q_*) = 0$ again,

$$\chi q_*\mathcal{G}'(q_*) = \rho\mathcal{G}(q_*) > 0. \tag{A.23}$$

The two inequalities (A.22) and (A.23) are in contradiction. ■

Figure A-1 plots $\mathcal{G}(q)$ relying on the parameter values in the legend. The remuneration policy consists in no payments made to the government until the government continuation utility q_t reaches the threshold q^o. The solution for $\mathcal{G}(q)$ for $q \leq q^o$ is calculated by expressing it in terms of confluent hypergeometric functions of the first kind, as explained by Mele (2022, Appendix 5.C). Instead, $\mathcal{G}(q) = \mathcal{G}(q^o) + q - q^o$ for $q \geq q^o$. The first best is $\mathcal{G}(q) = q$: when no incentives are needed to make governments behave, the citizens' costs are the same as the governments' continuation utility.

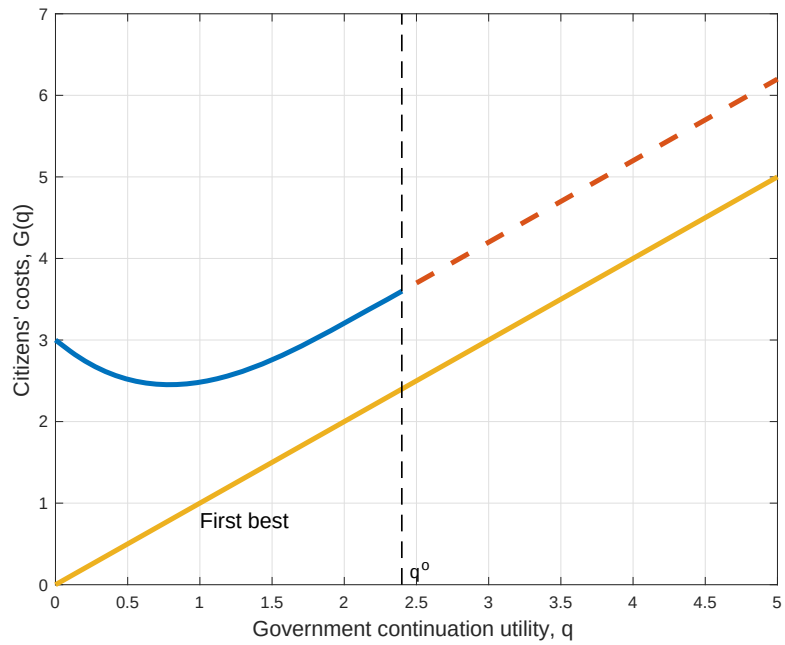

Figure A-1: Incentive-compatible mandates: Citizens' costs and governments' continuation utility. This picture depicts the solution to (A.21) (the upper curve) when the parameter values are $L = 3$, $\rho = 0.20$, $\chi = 0.30$ and, finally, the product $\psi\sigma\lambda = 0.8$. The first best (lower curve) is obtained as $\mathcal{G}(q) = q$.

Appendix B: Distortionary policies

B.1. Utility costs in the benchmark model

When governments do not internalize the effects of their deficit policy on growth and interest rates, the Bellman equation is as in (7), with $\kappa(\delta)$ replacing κ (see Section 3.6). The expression for the government utility costs is, then, still $V(\delta) = V_{\mathcal{D}}(\delta)\mathbf{1}_{\delta<\hat{\delta}} + V_{\mathcal{S}}(\delta)\mathbf{1}_{\delta>\hat{\delta}}$, where $V_{\mathcal{D}}(\cdot)$ and $V_{\mathcal{S}}(\cdot)$ are as in (A.3) but the coefficients $m_{\mathcal{X}j}$, for $\mathcal{X} = \mathcal{D}, \mathcal{S}$ and $j = 1, 2$, are

$$m_{\mathcal{X}1/\mathcal{X}2} = \frac{1}{2} + \frac{s_{\mathcal{X}} + \kappa_{\mathcal{X}}}{\sigma^2} \mp \sqrt{\left(\frac{1}{2} + \frac{s_{\mathcal{X}} + \kappa_{\mathcal{X}}}{\sigma^2}\right)^2 + \frac{2\rho}{\sigma^2}}, \quad s_{\mathcal{D}} \equiv s^1, \quad s_{\mathcal{S}} \equiv s^2,$$

where $\kappa_{\mathcal{D}} = \kappa^1$, $\kappa_{\mathcal{S}} = \kappa^2$, and

$$\kappa^j = \mu^j - i^j - \sigma^2, \quad j = 1, 2.$$

We assume that $\kappa^1 > \kappa^2$. Therefore, note that the debt limit in the model with exogenous default is the same as $\bar{\delta}$ in (8), but with κ^2 replacing κ, viz

$$\bar{\delta} = \frac{\ell e^{(s^1+\kappa^2-\bar{\iota})\bar{\varepsilon}}}{i+\bar{\iota}-s^1}. \tag{B.1}$$

Finally, note that conditions for the existence of a solution are that $Y_1 < 0$ and $Y_2 > 0$ in (A.9). As noted in Appendix A, these conditions are satisfied in the benchmark models. However, these conditions do not always hold in the model with distortionary policies, and the model solution is implemented while ensuring that $Y_1 < 0$ and $Y_2 > 0$ in (A.9).

Figures A-2 and A-3 depict utility costs and the level of fiscal tipping points in experiments where the growth rates of the economy in the deficit and surplus regimes differ. Figure A-2 also compares the value function with that of the model without distortionary policies. Figures A-4 and A-5 contain results from experiments where short-term interest rates in the deficit and surplus regimes differ. Except otherwise explained in the legends, parameter values are as in the legend of Figure 1.

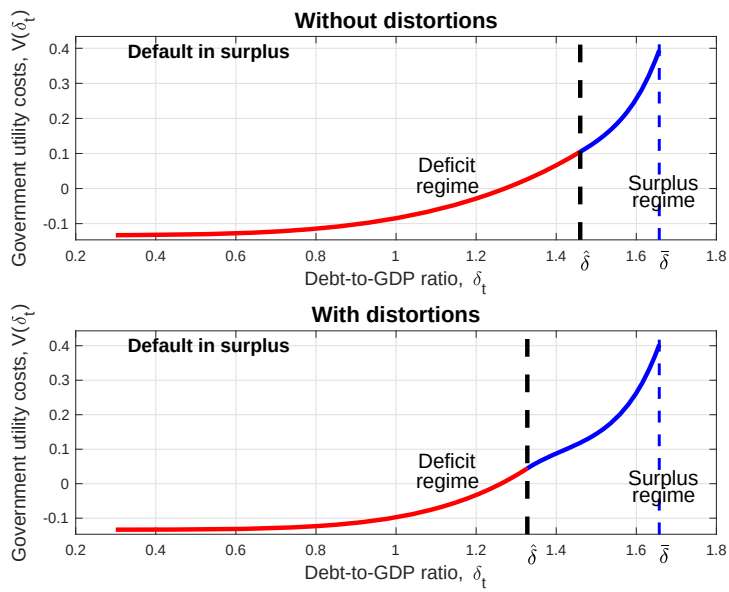

Figure A-2: Government utility costs with distortionary policies—Budget sizes and growth. Government utility costs in the model without (top panel) and with (bottom panel) distortionary policies. In the top panel, $\mu^1 = \mu^2 = -0.005$; in the bottom panel, $\left(\mu^1, \mu^2\right) = (0.015, -0.005)$. In both cases, budget sizes are fixed at $\left(s^1, s^2\right) = (-0.04, 0.03)$, and $\hat{\delta}$ and $\bar{\delta}$ indicate the fiscal tipping point and default boundary.

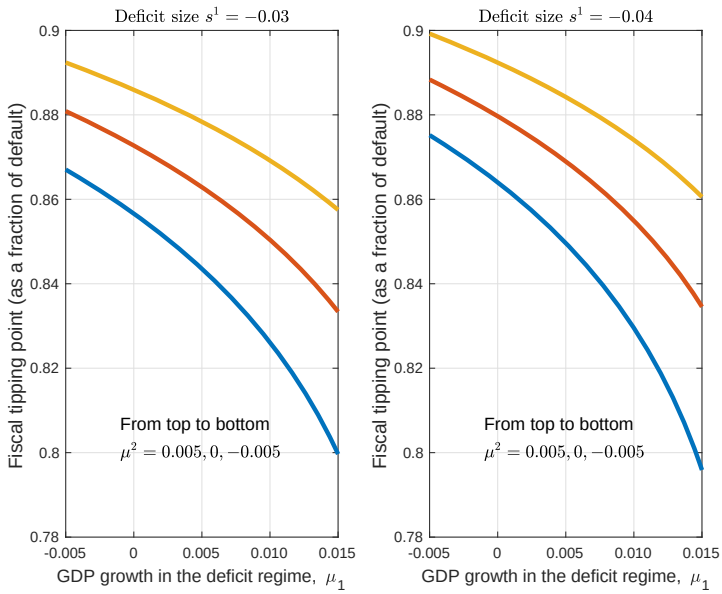

FIGURE A-3: FISCAL TIPPING POINTS AND DISTORTIONARY POLICIES—BUDGET SIZES AND GROWTH. This picture depicts the debt-to-GDP ratio $\hat{\delta}$ that triggers a change in the fiscal regime, expressed as a fraction of the default boundary, $C = \frac{\hat{\delta}}{\bar{\delta}}$, as a function of growth in the deficit regime, μ^1, under different assumptions on growth in the surplus regime. The left (resp., right) panel depicts fiscal tipping points for budget sizes fixed at $\left(s^1, s^2\right) = (-0.03, 0.03)$ (resp., $\left(s^1, s^2\right) = (-0.04, 0.03)$).

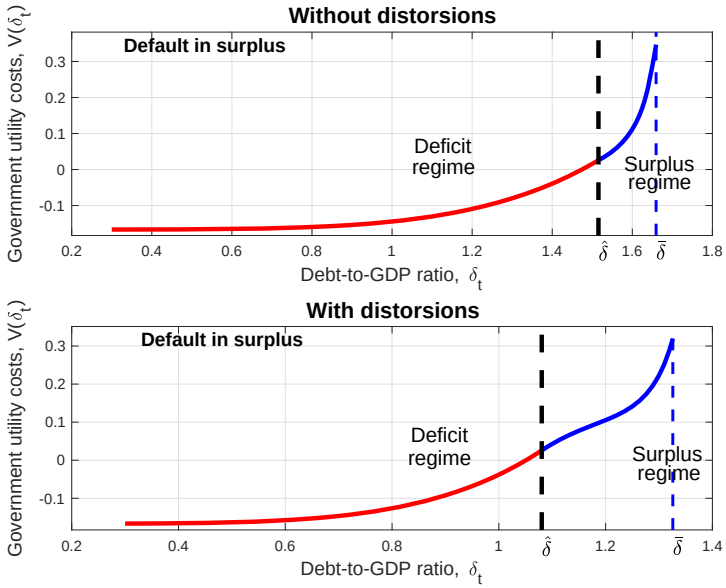

FIGURE A-4: GOVERNMENT UTILITY COSTS WITH DISTORTIONARY POLICIES—INTEREST RATES. Government utility costs in the model without (top panel) and with (bottom panel) distortionary policies. In the top panel, $i^1 = i^2 = 0.01$ and in the bottom panel, $\left(i^1, i^2\right) = (0.01, 0.025)$. In both cases, growth is fixed at $\mu^1 = \mu^2 = 0.01$, and $\hat{\delta}$ and $\bar{\delta}$ indicate the fiscal tipping point and default boundary.

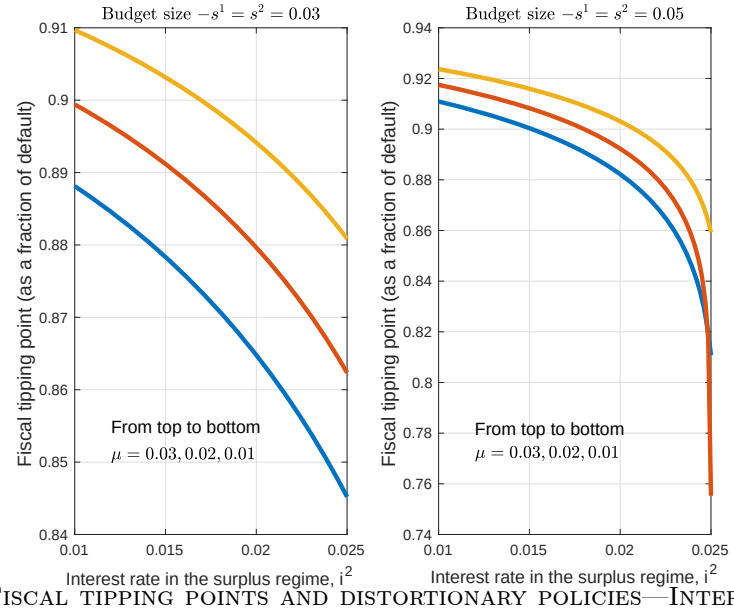

FIGURE A-5: FISCAL TIPPING POINTS AND DISTORTIONARY POLICIES—INTEREST RATES. This picture depicts the debt-to-GDP ratio $\hat{\delta}$ that triggers a change in the fiscal regime, expressed as a fraction of the default boundary, $C = \frac{\hat{\delta}}{\bar{\delta}}$, as a function of the short-term rate in the surplus regime, i^2, assuming that the short-term rate in the deficit regime is $i^1 = 0.01$, and under different assumptions on the growth rate of the economy. The left (resp., right) panel depicts fiscal tipping points for budget sizes fixed at $-s^1 = s^2 = 0.03$ (resp., $-s^1 = s^2 = 0.05$).

B.2. Policies with internalizations

We provide details on the model with internalization effects. The relevant Bellman equation is (11) (see Section 3.6), such that the governments' problem is to solve the following problem: for any given δ,

$$s_*(\delta) = \sup_s \left[s\left(\mathcal{V}(\delta) - 1\right) + \kappa(s)\,\mathcal{V}(\delta) \right], \tag{B.2}$$

where, by assumption, $\kappa(s) = \mathbf{1}_{s\leq 0}\kappa^1 + \mathbf{1}_{s>0}\kappa^2$ where the two constants κ^i are such that $\Delta\kappa \equiv \kappa^1 - \kappa^2$ satisfies the following condition:

Condition B.1. $\Delta\kappa \in \left(0, s^2\right)$.

Similarly as in Appendix A, we define the governments' marginal utility costs as $\mathcal{V}(\delta) = V'(\delta)\,\delta$. First, we conjecture that there are three regions for δ in which governments run a positive deficit, a zero deficit (the balanced regime), and positive surplus. Second, we formulate conditions under which we confirm this conjecture and characterize the three regions. Crucial for proving the conjecture is the proof that $\mathcal{V}'(\delta) > 0$, a property confirmed at the end of this appendix.

Three-region regimes. We conjecture and, later, verify that $\mathcal{V}(\delta) > 0$ and that $\mathcal{V}'(\delta) > 0$ for all δ. Consider, first, the case with small marginal costs, $\mathcal{V}(\delta) < 1$. Suppose that the solution to (B.2) is some $s_*(\delta) > 0$, such that, then, $\kappa(s) = \kappa^2$. Therefore, for any other candidate $s \leq 0$, such that, then, $\kappa(s) = \kappa^1$, we must have that

$$s_*(\delta)\left(\mathcal{V}(\delta) - 1\right) + \kappa^2\mathcal{V}(\delta) > s\left(\mathcal{V}(\delta) - 1\right) + \kappa^1\mathcal{V}(\delta),$$

which is impossible. Therefore, the solution to (B.2) must be some $s_*(\delta) \leq 0$, such that, then, $\kappa(s) = \kappa^1$ and it is trivial to verify that $s_*(\delta) = s^1$. Next, consider the case in which $\mathcal{V}(\delta) = 1$, such that $s_*(\delta) = \sup_s \kappa(s) = \{s : s \leq 0\}$. The solution is indeterminate: any deficit policy is optimal, but we assume that $s_*(\delta) = s^1$ in this case too.

Next, consider the case $\mathcal{V}(\delta) > 1$, define the function $\zeta(\delta) \equiv \left(\mathcal{V}(\delta) - 1\right)/\mathcal{V}(\delta)$ and two values of the debt-to-GDP ratio, $\hat{\delta}_1$ and $\hat{\delta}_2$, such that

$$\hat{\delta}_1 : \zeta(\hat{\delta}_1) = 0 \quad \text{and} \quad \hat{\delta}_2 : \zeta(\hat{\delta}_2) = \frac{\kappa^1 - \kappa^2}{s^2}. \tag{B.3}$$

Under our conjecture that $\mathcal{V}'(\delta) > 0$ (validated under conditions provided below), the function $\zeta(\cdot)$ is, then, also monotonically increasing for all $\delta \geq \hat{\delta}_1$, and we claim that the two values, $\hat{\delta}_1$ and $\hat{\delta}_2$ are the two thresholds of the optimal fiscal regime: a zero-deficit regime, $s_*(\delta) = 0$, for $\delta \in (\hat{\delta}_1, \hat{\delta}_2)$, and a surplus regime, with $s_*(\delta) = s^2$ for $\delta > \hat{\delta}_2$. Indeed, suppose the claim is correct and let $\delta \in (\hat{\delta}_1, \hat{\delta}_2)$, such that $s_*(\delta) = 0$ and, then, $s_*(\delta)\left(\mathcal{V}(\delta) - 1\right) + \kappa^1\mathcal{V}(\delta) = \kappa^1\mathcal{V}(\delta) > s^2\left(\mathcal{V}(\delta) - 1\right) + \kappa^2\mathcal{V}(\delta)$. By re-arranging the last inequality,

$$\zeta(\delta) \equiv \frac{\mathcal{V}(\delta) - 1}{\mathcal{V}(\delta)} < \frac{\kappa^1 - \kappa^2}{s^2}, \quad \text{for } \delta \in (\hat{\delta}_1, \hat{\delta}_2).$$

For $\delta > \hat{\delta}_2$, and under the claim that $s_*(\delta) = s^2$, we have $s_*(\delta)\left(\mathcal{V}(\delta) - 1\right) + \kappa^2\mathcal{V}(\delta) = s^2\left(\mathcal{V}(\delta) - 1\right) + \kappa^2\mathcal{V}(\delta) > s\left(\mathcal{V}(\delta) - 1\right)|_{s=0} + \kappa^1\mathcal{V}(\delta)$. By re-arranging the last inequality, and using the definition of $\zeta(\cdot)$,

$$\zeta(\delta) > \frac{\kappa^1 - \kappa^2}{s^2}, \quad \text{for } \delta > \hat{\delta}_2,$$

Therefore, the thresholds in (B.3) are the fiscal tipping points, provided $\zeta(\delta)$ is increasing. We now determine governments' utility costs and provide conditions under which $\mathcal{V}(\delta)$ and, then, $\zeta(\delta)$, are increasing.

Governments' utility costs. We conjecture that, under conditions developed below, the solution is as in Eq. (12) of the main text, where the three functions $V_{\mathcal{X}}(\delta)$, $\mathcal{X} \in (\mathcal{D}, \mathcal{B}, \mathcal{S})$ are given by

$$V_{\mathcal{D}}(\delta) = \frac{s^1}{\rho} + A_{\mathcal{D}2}\delta^{m_{\mathcal{D}2}}, \quad V_{\mathcal{B}}(\delta) = A_{\mathcal{B}1}\delta^{m_{\mathcal{B}1}} + A_{\mathcal{B}2}\delta^{m_{\mathcal{B}2}}, \quad V_{\mathcal{S}}(\delta) = \frac{s^2}{\rho} + A_{\mathcal{S}1}\delta^{m_{\mathcal{S}1}} + A_{\mathcal{S}2}\delta^{m_{\mathcal{S}2}}, \tag{B.4}$$

for some constants $A_{\mathcal{X}j}$ and $m_{\mathcal{X}j}$. Indeed, (B.4) follows by replacing Eq. (12) into the Bellman equation (11), and by the arguments in Appendix A, we have that $m_{\mathcal{D}2} > 0$, $m_{\mathcal{B}1} < 0$, $m_{\mathcal{B}2} > 0$, $m_{\mathcal{S}1} < 0$, $m_{\mathcal{S}2} > 0$. We now have to distinguish three cases that are relevant according to the region in which default occurs. Define the following constant

$$\bar{B} \equiv \frac{s^1}{\rho} + \frac{1}{m_{\mathcal{D}2}}. \tag{B.5}$$

We claim that default occurs in the deficit regime when $\bar{V} < \bar{B}$, in the balanced regime when $\bar{V} \in [\bar{B}, \psi]$ and, finally, in the surplus regime when $\bar{V} > \psi$, where ψ is a constant defined below (see (B.10)) that is strictly greater than $\bar{B}$ (see (B.12)).

- $\bar{V} < \bar{B}$: *Default in the deficit regime,* $\bar{\delta} < \hat{\delta}_1$. The solution is as in Appendix A, with $\hat{\delta}_1 = \hat{\delta}$ and $A_{\mathcal{D}2}$ given in (A.6). Note that this case occurs precisely when (A.7) holds.
- $\bar{V} \in [\bar{B}, \psi]$: *Default in the balanced regime,* $\bar{\delta} \in (\hat{\delta}_1, \hat{\delta}_2)$. The solution is still as in Appendix A (see Eq. (A.3)), with $\hat{\delta}_1 = \hat{\delta}$, but $s^2 = 0$, viz $V(\delta) = V_{\mathcal{D}}(\delta)\mathbf{1}_{\delta<\hat{\delta}_1} + V_{\mathcal{B}}(\delta)\mathbf{1}_{\delta>\hat{\delta}_1}$, where $V_{\mathcal{B}}(\delta)$ is as in (B.4) and the coefficients $m_{\mathcal{B}i}$ are obtained while setting $s^2 = 0$. Therefore, $A_{\mathcal{D}2}$ is as in (A.11), and $A_{\mathcal{B}i} = \bar{Y}_i\hat{\delta}_1^{-m_{\mathcal{B}i}}$, where

  $$A_{\mathcal{D}2} = \frac{\hat{\delta}_1^{-m_{\mathcal{D}2}}}{m_{\mathcal{D}2}}, \quad A_{\mathcal{B}i} = \bar{Y}_i\hat{\delta}_1^{-m_{\mathcal{B}i}}, \quad i = 1, 2, \tag{B.6}$$

  and

  $$\bar{Y}_1 = \frac{\bar{B}m_{\mathcal{B}2} - 1}{m_{\mathcal{B}2} - m_{\mathcal{B}1}}, \quad \bar{Y}_2 = \frac{1 - \bar{B}m_{\mathcal{B}1}}{m_{\mathcal{B}2} - m_{\mathcal{B}1}}, \tag{B.7}$$

  with $\bar{B}$ defined as in (B.5). The default boundary condition that is the counterpart to (A.10) is

  $$\bar{V} = \bar{Y}_1 x^{-m_{\mathcal{B}1}} + \bar{Y}_2 x^{-m_{\mathcal{B}2}}, \tag{B.8}$$

  where $x \equiv \hat{\delta}_1/\bar{\delta}$. By the same arguments in Appendix A, there is a unique solution to Eq. (B.8), $x(\bar{V}) < 1$, such that $\hat{\delta}_1 = x(\bar{V})\bar{\delta}$. Below, we show indeed that default cannot occur in the surplus regime when $\bar{V} < \psi$.
- $\bar{V} > \psi$: *Default in the surplus regime,* $\bar{\delta} > \hat{\delta}_2$. The governments' utility costs satisfy the boundary conditions (i)-(vii) below, which lead to determine the constants $A_{\mathcal{X}j}$. Define $\Omega \equiv \frac{s^2}{s^2 - \Delta\kappa}$. We shall need the following condition:

  Condition B.2. $\Omega > m_{\mathcal{S}2}(\psi - \frac{s^2}{\rho})$.

  We are now ready to determine all coefficients.

  (i) *Free boundary I:* $V'_{\mathcal{D}}(\delta)\,\delta|_{\delta=\hat{\delta}_1} = 1$. (ii) *Value matching I:* $V_{\mathcal{D}}(\hat{\delta}_1) = V_{\mathcal{B}}(\hat{\delta}_1)$. (iii) *Smooth pasting I:* $V'_{\mathcal{D}}(\delta)|_{\delta=\hat{\delta}_1} = V'_{\mathcal{B}}(\delta)|_{\delta=\hat{\delta}_1}$. By arguments nearly identical to those in Appendix A, the coefficients of $V_{\mathcal{D}}(\cdot)$ and $V_{\mathcal{B}}(\cdot)$ are as in (B.6)-(B.7), but with $\hat{\delta}_1$ is determined below.

  (iv) *Free boundary II:* The second condition in (B.3) implies that $V'_{\mathcal{B}}(\delta)\,\delta|_{\delta=\hat{\delta}_2} = \Omega > 1$, where the inequality follows by Condition B.1. Calculating the derivative and using the expressions of $A_{\mathcal{B}i}$ in (B.6) leaves

  $$\Psi(z) \equiv m_{\mathcal{B}1}\bar{Y}_1 z^{m_{\mathcal{B}1}} + m_{\mathcal{B}2}\bar{Y}_2 z^{m_{\mathcal{B}2}} = \Omega, \tag{B.9}$$

where $z \equiv \hat{\delta}_2/\hat{\delta}_1$. Now, note that the two constants $\bar{Y}_1$ and $\bar{Y}_2$ in (B.7)) coincide with Y_1 and Y_2 in (A.9) in the special case $s^2 = 0$. Therefore, by repeating the arguments in Appendix A and in the Internet Appendix A, we have that $\bar{Y}_1 < 0$ and $\bar{Y}_2 > 0$. It follows that $\Psi(z)$ has a unique stationary point for some $z > 0$, and that $\lim_{z \longrightarrow 0} \Psi(z) = \infty$, $\lim_{z \longrightarrow \infty} \Psi(z) = \infty$, and $\Psi(1) = m_{\mathcal{B}1}\bar{Y}_1 + m_{\mathcal{B}2}\bar{Y}_1 = 1$. Because $\Omega \geq 1$, Eq. (B.9) has, then, two roots, $z_1 < 1$ and $z_* > 1$, and we take the second solution, which guarantees that $\hat{\delta}_2 > \hat{\delta}_1$.

(*v*) *Value matching II:* $V_{\mathcal{B}}(\hat{\delta}_2) = V_{\mathcal{S}}(\hat{\delta}_2)$. (*vi*) *Smooth-pasting II:* $V'_{\mathcal{B}}(\delta)|_{\delta=\hat{\delta}_2} = V'_{\mathcal{S}}(\delta)|_{\delta=\hat{\delta}_2}$. By evaluating these conditions, using the expressions of $A_{\mathcal{B}i}$ in (B.6) and, finally, the definition of z_* leaves the following system of equations

$$\begin{cases} \psi \equiv \bar{Y}_1 z_*^{m_{\mathcal{B}1}} + \bar{Y}_2 z_*^{m_{\mathcal{B}2}} = \dfrac{s^2}{\rho} + A_{\mathcal{S}1}\hat{\delta}_2^{m_{\mathcal{S}1}} + A_{\mathcal{S}2}\hat{\delta}_2^{m_{\mathcal{S}2}} \\ \Omega = m_{\mathcal{B}1}\bar{Y}_1 z_*^{m_{\mathcal{B}1}} + m_{\mathcal{B}2}\bar{Y}_2 z_*^{m_{\mathcal{B}2}} = m_{\mathcal{S}1}A_{\mathcal{S}1}\hat{\delta}_2^{m_{\mathcal{S}1}} + m_{\mathcal{S}2}A_{\mathcal{S}2}\hat{\delta}_2^{m_{\mathcal{S}2}} \end{cases} \tag{B.10}$$

where the first equality of the second equation follows by (B.9). The solution is

$$A_{\mathcal{S}1} = \frac{m_{\mathcal{S}2}\left(\psi - \dfrac{s^2}{\rho}\right) - \Omega}{m_{\mathcal{S}2} - m_{\mathcal{S}1}}\hat{\delta}_2^{-m_{\mathcal{S}1}}, \quad A_{\mathcal{S}2} = \frac{\Omega - m_{\mathcal{S}1}\left(\psi - \dfrac{s^2}{\rho}\right)}{m_{\mathcal{S}2} - m_{\mathcal{S}1}}\hat{\delta}_2^{-m_{\mathcal{S}2}}. \tag{B.11}$$

It is straightforward to show that

$$\psi > \bar{Y}_1 + \bar{Y}_2 = \bar{B}. \tag{B.12}$$

Next, we determine $\hat{\delta}_2$, through the default boundary condition (*vii*) below.

(*vii*) *Default boundary:* $\bar{V} = V_{\mathcal{S}}(\bar{\delta})$, that is,

$$\bar{V} = \frac{s^2}{\rho} + A_{\mathcal{S}1}\bar{\delta}^{m_{\mathcal{S}1}} + A_{\mathcal{S}2}\bar{\delta}^{m_{\mathcal{S}2}}. \tag{B.13}$$

Plugging the expressions for $A_{\mathcal{S}i}$ in (B.11) into (B.13) leaves a nonlinear equation in $\hat{\delta}_2$. Note that $m_{\mathcal{S}1} < 0$ and $m_{\mathcal{S}2} > 0$. We claim that $A_{\mathcal{S}2} > 0$. Indeed, note that $M \equiv m_{\mathcal{S}1}(\psi - \frac{s^2}{\rho}) < m_{\mathcal{S}1}(\bar{B} - \frac{s^2}{\rho}) = m_{\mathcal{S}1}B$, where the inequality follows by the fact that $m_{\mathcal{S}1}$ and by (B.12), and the equality follows by the definition of B in (A.9) and $\bar{B}$ in (B.5). Now, if $B > 0$, $M < 0$, such that $A_{\mathcal{S}2} > 0$; if $B < 0$, then, the Internet Appendix A shows that $m_{\mathcal{S}1}B < 1$, such that $\Omega > 1 > m_{\mathcal{S}1}B > M$ and, thus, $A_{\mathcal{S}2} > 0$. Therefore, we have that $A_{\mathcal{S}2} > 0$ and, under Condition B.2, $A_{\mathcal{S}1} < 0$. There exists, then, a unique solution for $\hat{\delta}_2$. We claim that this solution satisfies $\hat{\delta}_2 < \bar{\delta}$. Indeed, denote $A_{\mathcal{S}i} \equiv a_{\mathcal{S}i}\hat{\delta}_2^{-m_{\mathcal{S}i}}$, for two constants $a_{\mathcal{S}i}$ that obviously satisfy $a_{\mathcal{S}1} < 0$ (under Condition B.2) and $a_{\mathcal{S}2} > 0$, such that Eq. (B.13) can be written as

$$\bar{V} - \frac{s^2}{\rho} = g(u) \equiv a_{\mathcal{S}1}u^{m_{\mathcal{S}1}} + a_{\mathcal{S}2}u^{m_{\mathcal{S}2}}, \tag{B.14}$$

where $u \equiv \bar{\delta}/\hat{\delta}_2$. Note that $g(1) = \psi - \frac{s^2}{\rho}$ and, since g is increasing, the solution to (B.14) is some $u^* > 1$ if and only if $\psi < \bar{V}$.

To summarize, given the solution for $\hat{\delta}_2$, $A_{\mathcal{S}i}$ is determined through (B.11), $\hat{\delta}_1$ is determined as $\hat{\delta}_1 = \hat{\delta}_2/z_*$, and, then, $A_{\mathcal{D}2}$ and $A_{\mathcal{B}i}$ are determined through (B.6).

Finally, we show that, under all conditions formulated so far, $\mathcal{V}(\delta)$ and, then, $\zeta(\delta)$, are increasing in all cases.

Monotonicity of marginal utility costs. The property that $\mathcal{V}(\delta)$ is increasing follows by the assumptions in this appendix and the same arguments used in Appendix A to show that government's marginal utility costs are strictly positive.

Appendix C: Strategic default

We verify the conjectures in Section 3.7 by proceeding as follows. First, we search for a fiscal tipping point by imposing the same conditions applying to the simpler problem (2) in the main text. Second, we impose free-boundary conditions and determine the default boundary, a constant $\bar{\delta}_o$ indeed. Third, we compare $\bar{\delta}_o$ with $\bar{\delta}$, the exogenous default boundary in the previous sections: if $\bar{\delta}_o < \bar{\delta}$, the government defaults at $\bar{\delta}_o$; otherwise, the model details are the same as those of the model with exogenous default.

The next proposition confirms that governments may indeed default before δ_t hits the exogenous default trigger in the model with exogenous default. We have:

Proposition C.1. (Default boundaries). *The government utility costs are given by the same function $V(\delta_t)$ in Proposition I, but subject to boundary conditions provided below (see (C.4)). Under conditions provided below (see Lemma C.1), the government policy is to default at some finite $\bar{\delta}_o$, provided $\bar{\delta}_o < \bar{\delta}$, where $\bar{\delta}$ is the exogenous default boundary in Eq. (8), or to still remain in office until δ_t reaches $\bar{\delta}$. There exist parameter values such that $\bar{\delta}_o < \bar{\delta}$.*

By convention, we take $\bar{\delta}_o = \infty$ whenever $\bar{\delta}_o$ does not satisfy the verification conditions in the previous proposition. We now prove Proposition C.1. The Internet Appendix C provides numerical results on default boundaries and fiscal tipping points and additional comparative statics results for this model.

We begin by stating a necessary condition for governments' default in the deficit regime.

Condition C.1. $s^1 + \kappa + \rho < 0$.

Next, we determine policy regimes and default boundaries.

Government utility costs, fiscal tipping points and default boundaries. We rely on the same notation employed while dealing with government utility costs in the exogenous default case (see Eq. (2)). Whilst the government runs an active policy, the function V in Eq. (13) still satisfies Eq. (A.1), as well as the boundary conditions (i)-(ii)-(iii) in (A.8), and the general solution for the utility costs is still as in Appendix A, with $A_{\mathcal{D}1} = 0$, i.e., $V(\delta) = V_{\mathcal{D}}(\delta)\mathbf{1}_{\delta<\hat{\delta}} + V_{\mathcal{S}}(\delta)\mathbf{1}_{\delta>\hat{\delta}}$, where

$$V_{\mathcal{D}}(\delta) = \frac{s^1}{\rho} + A_{\mathcal{D}2}\delta^{m_{\mathcal{D}2}}, \quad V_{\mathcal{S}}(\delta) = \frac{s^2}{\rho} + A_{\mathcal{S}1}\delta^{m_{\mathcal{S}1}} + A_{\mathcal{S}2}\delta^{m_{\mathcal{S}2}}. \tag{C.1}$$

However, the boundary behavior of $V_{\mathcal{D}}(\delta)$ and $V_{\mathcal{S}}(\delta)$ differs from that in Appendix A, because the default boundary $\bar{\delta}_o$ is, now, endogenous. We proceed as follows. Note that the value matching condition and the smooth pasting conditions are

$$V(\bar{\delta}_o) = \mathcal{C}(\bar{\delta}_o), \quad V'(\bar{\delta}_o) = \mathcal{C}'(\bar{\delta}_o), \tag{C.2}$$

where, by the assumption that re-entry occurs in the deficit regime, the government utility costs of defaulting are as in Eq. (10); that is, evaluating $\mathcal{C}(\delta)$ at $\bar{\delta}_o$,

$$\mathcal{C}(\bar{\delta}_o) = \xi\bar{\delta}_o + \frac{\epsilon}{\rho+\vartheta} + \frac{\vartheta}{\rho+\vartheta}\left(\frac{s^1}{\rho} + A_{\mathcal{D}2}\left(\gamma\bar{\delta}_o\right)^{m_{\mathcal{D}2}}\right). \tag{C.3}$$

We apply the value matching and the smooth pasting conditions in (C.2) based on the expression of $\mathcal{C}(\bar{\delta}_o)$ in Eq. (C.3). We consider two cases, arising according to whether default occurs in the deficit or in the surplus regime. In each case, the value matching condition and the smooth pasting conditions are, respectively,

$$V_{\mathcal{X}}(\bar{\delta}_o) = \mathcal{C}(\bar{\delta}_o), \quad V'_{\mathcal{X}}(\bar{\delta}_o) = \mathcal{C}'(\bar{\delta}_o), \quad \mathcal{X} = \{\mathcal{D}, \mathcal{S}\}, \tag{C.4}$$

where the two functions $V_{\mathcal{D}}(\delta)$ and $V_{\mathcal{S}}(\delta)$ are as in Eq. (C.1), and $\mathcal{C}(\delta)$ is as in Eq. (C.3).

- *Default in the deficit regime.* Assume that default occurs in the deficit regime. For $\mathcal{X} = \mathcal{D}$, the two conditions in (C.4) lead to a system of two equations with two unknowns, $\bar{\delta}_{\mathcal{D}}$ (say) and $A_{\mathcal{D}2}$, solved by

$$\bar{\delta}_{\mathcal{D}} \equiv \frac{m_{\mathcal{D}2}}{1 - m_{\mathcal{D}2}} \frac{\epsilon - s^1}{\xi(\rho + \vartheta)}, \quad A_{\mathcal{D}2} = \frac{\xi(\rho + \vartheta)}{m_{\mathcal{D}2}(\rho + (1 - \gamma^{m_{\mathcal{D}2}})\vartheta)} \bar{\delta}_{\mathcal{D}}^{1 - m_{\mathcal{D}2}}. \tag{C.5}$$

  Under Condition B.1, $m_{\mathcal{D}2} < 1$, and, thus, the solution for $\bar{\delta}_{\mathcal{D}}$ is well-defined. Next, we verify that the tipping point, $\hat{\delta}_{\mathcal{D}}$ (say), is larger than the default boundary. The solution for the tipping point is determined while only applying the free boundary condition to the utility costs in the deficit regime, i.e., $V'_{\mathcal{D}}(\delta)\,\delta|_{\delta=\hat{\delta}} = 1$, which yields the following solution for the fiscal tipping point

$$\hat{\delta}_{\mathcal{D}} \equiv \left(\frac{1}{m_{\mathcal{D}2} A_{\mathcal{D}2}}\right)^{\frac{1}{m_{\mathcal{D}2}}}. \tag{C.6}$$

  Governments choose to default in the deficit regime if $\bar{\delta}_{\mathcal{D}} \in (0, \hat{\delta}_{\mathcal{D}})$. If $\bar{\delta}_{\mathcal{D}} \notin (0, \hat{\delta}_{\mathcal{D}})$, and default does not, then, occur in the deficit regime, governments may choose to default in the surplus regime, a case analyzed next.

- *Default in the surplus regime.* For $\mathcal{X} = \mathcal{S}$, the value matching and smooth pasting conditions in (C.4) are, respectively,

$$A_{\mathcal{S}1}\bar{\delta}_o^{m_{\mathcal{S}1}} + A_{\mathcal{S}2}\bar{\delta}_o^{m_{\mathcal{S}2}} - \frac{\vartheta}{\rho + \vartheta} A_{\mathcal{D}2}\gamma^{m_{\mathcal{D}2}}\bar{\delta}_o^{m_{\mathcal{D}2}} = \xi\bar{\delta}_o + \frac{\rho\left(\epsilon - s^2\right) + \vartheta\left(s^1 - s^2\right)}{\rho(\rho + \vartheta)}, \tag{C.7}$$

  and

$$m_{\mathcal{S}1}A_{\mathcal{S}1}\bar{\delta}_o^{m_{\mathcal{S}1}} + m_{\mathcal{S}2}A_{\mathcal{S}2}\bar{\delta}_o^{m_{\mathcal{S}2}} - \frac{\vartheta}{\rho + \vartheta} m_{\mathcal{D}2}A_{\mathcal{D}2}\gamma^{m_{\mathcal{D}2}}\bar{\delta}_o^{m_{\mathcal{D}2}} = \xi\bar{\delta}_o, \tag{C.8}$$

  where $A_{\mathcal{D}2}$ and $A_{\mathcal{S}i}$ are as in (A.11), but with $\hat{\delta}$ to be determined below, not by Eq. (A.10). Replacing the expressions for $A_{\mathcal{D}2}$ and $A_{\mathcal{S}i}$ in (A.11) into (C.7)-(C.8), and re-arranging terms, leaves the following equation satisfied by $x \equiv \hat{\delta}/\bar{\delta}_o$,

$$(1 - m_{\mathcal{S}1})Y_1 x^{-m_{\mathcal{S}1}} + (1 - m_{\mathcal{S}2})Y_2 x^{-m_{\mathcal{S}2}} - \frac{\vartheta\gamma^{m_{\mathcal{D}2}}}{\rho + \vartheta}\frac{1 - m_{\mathcal{D}2}}{m_{\mathcal{D}2}} x^{-m_{\mathcal{D}2}} = \frac{\rho\left(\epsilon - s^2\right) + \vartheta\left(s^1 - s^2\right)}{\rho(\rho + \vartheta)}. \tag{C.9}$$

  Therefore, the solution for the fiscal tipping point and the default boundary are

$$\hat{\delta}_{\mathcal{S}} \equiv x_* \bar{\delta}_{\mathcal{S}}, \quad \bar{\delta}_{\mathcal{S}} = \xi^{-1}\left(m_{\mathcal{S}1}Y_1 x_*^{-m_{\mathcal{S}1}} + m_{\mathcal{S}2}Y_2 x_*^{-m_{\mathcal{S}2}} - \frac{\vartheta\gamma^{m_{\mathcal{D}2}}}{\rho + \vartheta} x_*^{-m_{\mathcal{D}2}}\right), \tag{C.10}$$

  where x_* is solution to (C.9), and the expression for $\bar{\delta}_{\mathcal{S}}$ follows by (C.8) and, again, the expressions for $A_{\mathcal{D}2}$ and $A_{\mathcal{S}i}$ in (A.11).

Thus, and assuming that governments default at some value lower than the $\bar{\delta}$, the exogenous default boundary, the solution for the utility costs in the deficit and surplus regimes is as in (C.1), and default

boundaries, tipping points and coefficients are as follows:

- If governments default in the deficit regime, the default boundary $\bar{\delta}_{\mathcal{D}}$ and the coefficient $A_{\mathcal{D}2}$ are given by (C.5).
- If governments default in the surplus regime, the fiscal tipping point $\hat{\delta}_{\mathcal{S}}$ and default boundary $\bar{\delta}_{\mathcal{S}}$ are given by (C.10), and the coefficients $A_{\mathcal{D}2}$ and $A_{\mathcal{S}i}$ are as in (A.11).

Finally, we show that, under parameter restrictions, default always occurs in the deficit regime, and this default boundary is less than $\bar{\delta}$, exogenous default boundary. Indeed, note that, by (C.5) and (C.6), we have that $\hat{\delta}_{\mathcal{D}} > \bar{\delta}_{\mathcal{D}}$ if and only if

$$\frac{\rho + (1 - \gamma^{m_{\mathcal{D}2}})\,\vartheta}{\epsilon - s^1} > \frac{m_{\mathcal{D}2}}{1 - m_{\mathcal{D}2}}.$$

Hence, there exist values of ϑ, γ and ϵ such that $\bar{\delta}_{\mathcal{D}} < \hat{\delta}_{\mathcal{D}}$, and values of the initial debt-to-GDP ratio, δ_0, such that $\delta_0 < \bar{\delta}_{\mathcal{D}}$. By Eq. (8), there exist parameter values such that $\bar{\delta}_{\mathcal{D}} < \bar{\delta}$. Note that the same property applies to $\bar{\delta}_{\mathrm{n}}$, the NPDL in Eq. (9).

Verification. Consider the program in (13),

$$V(\delta_t) = \inf_{\tau} E_t \left[\int_t^{\tau} e^{-\rho(u-t)} s(\delta_u)\, du + e^{-\rho(\tau - t)} \mathcal{C}(\delta_\tau) \right],$$

where $s(\delta)$ is the surplus policy held prior to default. Under conditions developed below, the value, V, is the unique viscosity solution to the Hamilton-Jacobi-Bellman equation

$$0 = \max\{-(Lw - \rho w + s), w - \mathcal{C}\},$$

or

$$\begin{cases} Lw - \rho w + s = 0 & \text{and} \quad w < \mathcal{C} \quad \text{(continuation region)} \\ Lw - \rho w + s \geq 0 & \text{and} \quad w = \mathcal{C} \quad \text{(default region)} \end{cases} \tag{C.11}$$

Indeed, by Itô's lemma,

$$\begin{aligned} & e^{-\rho(\tau-t)} V(\delta_\tau) + \int_t^{\tau} e^{-\rho(u-t)} s(\delta_u)\, du - V(\delta_t) \\ &= \int_t^{\tau} e^{-\rho(u-t)} \left(LV(\delta_u) - \rho V(\delta_u) + s(\delta_u) \right) du + \sigma \int_t^{\tau} e^{-\rho(u-t)} V'(\delta_u)\, \delta_u dW_u \\ &\geq \sigma \int_t^{\tau} e^{-\rho(u-t)} V'(\delta_u)\, \delta_u dW_u, \end{aligned}$$

where the last equality follows by (C.11). That is, for all τ,

$$V(\delta_t) \leq E_t \left[\int_t^{\tau} e^{-\rho(u-t)} s(\delta_u)\, du + e^{-\rho(\tau-t)} V(\delta_\tau) \right].$$

Therefore, we need to verify that (C.11) holds under suitable conditions.

Consider the following conditions:

Condition C.2. *For each* $i = 1, 2$,

$$-\left(s^i + \kappa + \rho\right) \xi \bar{\delta}_o \geq \frac{\rho}{\rho + \vartheta} \left(\epsilon - s^i\right) - \mathbf{1}_{i=2} \left(s^2 - s^1\right). \tag{C.12}$$

Moreover, there exists an arbitrarily small $\delta_* : V(\delta) \leq \mathcal{C}(\delta)$ *for all* $\delta \in \left(\delta_*, \bar{\delta}_o\right)$

We have:

Lemma C.1. *Suppose that Conditions C.1 and C2 hold. Then, there exist values of* $\gamma_o > 0$ *and* $\rho_o > 0$*, such that, if governments default in their deficit regime, (C.11) holds for all* $(\gamma, \rho) \in (0, \gamma_o) \times (0, \rho_o)$ *and, if governments default in their surplus regime, (C.11) holds for all* $\gamma \in (0, \gamma_o)$*.*

Proof. We verify (C.11) in both the continuation and default region.

- *Continuation region.* By construction, the function $V(\delta)$ satisfies the ordinary differential equation in the first line of (C.11). As for the inequality $V \leq \mathcal{C}$, note that V and $\mathcal{C}$ are both increasing. Therefore, by the value matching and smooth pasting conditions in (C.2), a necessary condition under which $V(\delta) \leq \mathcal{C}(\delta)$ for all $\delta \leq \bar{\delta}_o$ is that there exists an arbitrarily small δ_0 such that $V(\delta_0) < \mathcal{C}(\delta_0)$. But, $\lim_{\delta \longrightarrow 0} V(\delta) = \frac{s^1}{\rho} < 0$ and $\lim_{\delta \longrightarrow 0} \mathcal{C}(\delta) = \frac{\rho\epsilon + \vartheta s^1}{\rho(\rho+\vartheta)} > 0$, implying, by continuity, that such a δ_0 exists.
- *Default region.* Substituting $w = \mathcal{C}$ into the inequality in the second line of (C.11) leaves

$$L\mathcal{C}(\delta) - \rho\mathcal{C}(\delta) + s(\delta) \geq 0, \tag{C.13}$$

  where $\mathcal{C}(\delta)$ is as in Eq. (10). Substituting $\mathcal{C}(\delta)$, $\mathcal{C}'(\delta)$ and $\mathcal{C}''(\delta)$ from (10) into (C.13) leaves the following condition

$$s(\delta) - \frac{\rho\epsilon}{\rho+\vartheta} - (s(\delta) + \kappa + \rho)\xi\delta + \frac{\vartheta}{\rho+\vartheta}\left[-(s(\delta)+\kappa)\gamma\delta V'(\gamma\delta) + \frac{1}{2}\sigma^2(\gamma\delta)^2 V''(\gamma\delta) - \rho V(\gamma\delta)\right] \geq 0. \tag{C.14}$$

  Taking the limits for $\gamma \longrightarrow 0$ of both sides of (C.14), and by the assumption that governments' re-entry takes place in the deficit region, leaves the inequality (C.12) of Condition C.2. By Condition C.1, the L.H.S. of (C.12) is strictly positive. Therefore, suppose default occurs in the deficit region, i.e., $i = 1$ in Condition C.1; then, there exist values of $\rho_o > 0$, such that (C.12) holds for all $\rho \in (0, \rho_o)$. Instead, default occurs in the surplus region under Condition C.2 for $i = 2$.

Note that Proposition C.1 provides conditions for the existence of a default boundary; the inequality in (C.14) for $\delta = \bar{\delta}_o$ imposes less stringent parameter restrictions than those underlying (C.12).

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A Theory of Debt Accumulation and Deficit Cycles

Internet Appendix

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A. Additional proofs

A.1.

We provide details omitted from Appendix A, regarding the cases where government utility costs at default, $\bar{V}$, are both lower (Case I) and higher (Case II) than $\frac{s^2}{\rho}$.

Case I: $\bar{V} < \frac{s^2}{\rho}$
We show that $Y_2 > 0$. In Appendix A, we have already shown that $m_{\mathcal{S}1} < 0$ and $B < 0$. Therefore, it suffices to show that $(-B)(-m_{\mathcal{S}1}) < 1$, or, equivalently, that

$$\frac{s^2 - s^1}{\rho} - \frac{1}{m_{\mathcal{D}2}} < \frac{1}{-m_{\mathcal{S}1}} \tag{I-1}$$

$\Longleftrightarrow$

$$\begin{aligned}
\frac{s^2 - s^1}{\rho} &< \frac{1}{\left(\frac{1}{2} + \frac{s^1+\kappa}{\sigma^2}\right) + \sqrt{\left(\frac{1}{2} + \frac{s^1+\kappa}{\sigma^2}\right)^2 + \frac{2\rho}{\sigma^2}}} + \frac{1}{-\left(\frac{1}{2} + \frac{s^2+\kappa}{\sigma^2}\right) + \sqrt{\left(\frac{1}{2} + \frac{s^2+\kappa}{\sigma^2}\right)^2 + \frac{2\rho}{\sigma^2}}} \\
&\equiv \frac{1}{a_1 + \sqrt{a_1^2 + \frac{2\rho}{\sigma^2}}} + \frac{1}{-a_2 + \sqrt{a_2^2 + \frac{2\rho}{\sigma^2}}} \\
&= \frac{1}{2}\frac{\sigma^2}{\rho}\left(a_2 - a_1 + \sqrt{a_1^2 + \frac{2\rho}{\sigma^2}} + \sqrt{a_2^2 + \frac{2\rho}{\sigma^2}}\right) \\
&= \frac{s^2 - s^1}{\rho} + \frac{1}{2}\frac{\sigma^2}{\rho}\left(\sum_{j=1}^{2}\sqrt{a_j^2 + \frac{2\rho}{\sigma^2}} - (a_2 - a_1)\right),
\end{aligned}$$

where

$$a_j \equiv \frac{1}{2} + \frac{s^j + \kappa}{\sigma^2}, \quad j = 1, 2,$$

and the third line follows by

$$\frac{1}{A + \sqrt{A^2 + C}} + \frac{1}{-B + \sqrt{B^2 + C}} = \frac{1}{C}\left(B - A + \sqrt{A^2 + C} + \sqrt{B^2 + C}\right).$$

The result follows by the following inequalities

$$a_2 - a_1 < |a_2| + |a_1| < \sum_{j=1}^{2} \sqrt{a_j^2 + \frac{2\rho}{\sigma^2}}.$$

Case II: $\bar{V} > \frac{s^2}{\rho}$

There are two cases to consider, according to whether $B < 0$ or $B > 0$.

- <u>First case: $B < 0$</u>. If $B < 0$, we have that $Y_1 < 0$, because $m_{\mathcal{S}2} > 0$. Moreover, we also have that $Y_2 > 0$, by the same reasoning made in *Case I*. Therefore, the expressions for the fiscal tipping point are the same as those in *Case I*. Note, also, that in this case, $m_{\mathcal{D}2}(\bar{V} - \frac{s^1}{\rho}) > m_{\mathcal{D}2}(\frac{s^2 - s^1}{\rho}) > 1$, where the last equality follows by the assumption that $B < 0$. That is, (A.7) does not hold, and default can only occur in the surplus regime.
- <u>Second case: $B > 0$</u>. We prove that $Y_2 > 0$ and $Y_1 < 0$, such that the expressions for the fiscal tipping point are the same as those resulting from the parameter restriction in *Case I*.
  - *Proof that* $Y_2 > 0$. It suffices to show that $B > \frac{-1}{-m_{\mathcal{S}1}}$, or

    $$\frac{s^1 - s^2}{\rho} + \frac{1}{m_{\mathcal{D}2}} > \frac{-1}{-m_{\mathcal{S}1}}.$$

    But this is inequality (I-1).
  - *Proof that* $Y_1 < 0$. It suffices to show that $Bm_{\mathcal{S}2} < 1$, or

    $$\frac{s^1 - s^2}{\rho} + \frac{1}{m_{\mathcal{D}2}} < \frac{1}{m_{\mathcal{S}2}}$$

    $\Longleftrightarrow$

    $$\begin{aligned}
    \frac{s^1 - s^2}{\rho} &< -\frac{1}{\left(\frac{1}{2} + \frac{s^1+\kappa}{\sigma^2}\right) + \sqrt{\left(\frac{1}{2} + \frac{s^1+\kappa}{\sigma^2}\right)^2 + \frac{2\rho}{\sigma^2}}} + \frac{1}{\left(\frac{1}{2} + \frac{s^2+\kappa}{\sigma^2}\right) + \sqrt{\left(\frac{1}{2} + \frac{s^2+\kappa}{\sigma^2}\right)^2 + \frac{2\rho}{\sigma^2}}} \\
    &\equiv -\frac{1}{a_1 + \sqrt{a_1^2 + \frac{2\rho}{\sigma^2}}} + \frac{1}{a_2 + \sqrt{a_2^2 + \frac{2\rho}{\sigma^2}}} \\
    &= \frac{1}{2}\frac{\sigma^2}{\rho}\left(a_1 - a_2 - \sqrt{a_1^2 + \frac{2\rho}{\sigma^2}} + \sqrt{a_2^2 + \frac{2\rho}{\sigma^2}}\right) \\
    &= \frac{s^1 - s^2}{\rho} + \frac{1}{2}\frac{\sigma^2}{\rho}\left(-\sqrt{a_1^2 + \frac{2\rho}{\sigma^2}} + \sqrt{a_2^2 + \frac{2\rho}{\sigma^2}} - (a_1 - a_2)\right), \qquad \text{(I-2)}
    \end{aligned}$$

    where the third line follows by

    $$-\frac{1}{A + \sqrt{A^2 + C}} + \frac{1}{B + \sqrt{B^2 + C}} = \frac{1}{C}\left(A - B - \sqrt{A^2 + C} + \sqrt{B^2 + C}\right).$$

    Therefore, $Bm_{\mathcal{S}2} < 1$ if and only if the second term on the R.H.S of the inequality (I-2) is positive. Equivalently, let $x \equiv a_1 - a_2$, and define the function

    $$Z(x) \equiv \sqrt{a_2^2 + \frac{2\rho}{\sigma^2}} - \sqrt{(a_2 + x)^2 + \frac{2\rho}{\sigma^2}} - x.$$

By the assumption that $s^1 < 0$ and $s^2 > 0$, the proof is complete if we show that $Z > 0$ for all $x < 0$. Now, we have that $Z(0) = 0$, and, for all $x < 0$,

$$Z'(x) = -\frac{\sqrt{(a_2 + x)^2 + \frac{2\rho}{\sigma^2}} + a_2 + x}{\sqrt{(a_2 + x)^2 + \frac{2\rho}{\sigma^2}}} < 0.$$

That is, Z is monotonically decreasing, and reaches zero for $x = 0$, such that $Z(x) > 0$, for all $x < 0$. Thus, $Y_1 < 0$.

Thus, the function $\varphi(\cdot)$ in (A.10) is decreasing with $\lim_{x \longrightarrow 0} \varphi(x) = \infty$ and $\lim_{x \longrightarrow \infty} \varphi(x) = -\infty$. Note, then, that (A.7) holds if and only if governments default in the deficit regime. Indeed, recall the definition of $x(\bar{V})$ in Appendix A, and suppose that $x(\bar{V}) > 1$ (default in the deficit regime). Then, because $\varphi(\cdot)$ is decreasing, we have

$$\bar{V} - \frac{s^2}{\rho} = \varphi\left(x(\bar{V})\right) < \varphi(1) = Y_1 + Y_2 = B = \frac{s^1 - s^2}{\rho} + \frac{1}{m_{\mathcal{D}2}},$$

that is, condition (A.7) holds.

A.2.

We show that the inequality (A.7) in Appendix A holds if and only if the solution of (A.15) in the model with re-entry satisfies $x_{\mathcal{D}} > 1$. Suppose, indeed, that the inequality (A.7) holds. Then,

$$\begin{aligned}
\frac{1}{m_{\mathcal{D}2}} &> \bar{V} - \frac{s^1}{\rho} \\
&= \frac{\epsilon}{\rho + \vartheta} + \frac{\vartheta}{\rho + \vartheta}\left(\frac{s^1}{\rho} + \frac{\gamma^{m_{\mathcal{D}2}}}{m_{\mathcal{D}2}} x_{\mathcal{D}}^{-m_{\mathcal{D}2}}\right) + \xi\bar{\delta} - \frac{s^1}{\rho} \\
&= \frac{\epsilon - s^1}{\rho + \vartheta} + \frac{\vartheta}{\rho + \vartheta}\frac{\gamma^{m_{\mathcal{D}2}}}{m_{\mathcal{D}2}} x_{\mathcal{D}}^{-m_{\mathcal{D}2}} + \xi\bar{\delta}.
\end{aligned}$$

That is, by re-arranging terms,

$$\xi\bar{\delta} + \frac{\epsilon - s^1}{\rho + \vartheta} < \frac{\rho + \vartheta\left(1 - \gamma^{m_{\mathcal{D}2}} x_{\mathcal{D}}^{-m_{\mathcal{D}2}}\right)}{m_{\mathcal{D}2}(\rho + \vartheta)}.$$

Therefore, by Eq. (A.15), it must hold that

$$\frac{\rho + \vartheta\left(1 - \gamma^{m_{\mathcal{D}2}}\right)}{m_{\mathcal{D}2}(\rho + \vartheta)} x_{\mathcal{D}}^{-m_{\mathcal{D}2}} < \frac{\rho + \vartheta\left(1 - \gamma^{m_{\mathcal{D}2}} x_{\mathcal{D}}^{-m_{\mathcal{D}2}}\right)}{m_{\mathcal{D}2}(\rho + \vartheta)}.$$

It is easily verified that this inequality holds when $(\rho + \vartheta)\left(x_{\mathcal{D}}^{-m_{\mathcal{D}2}} - 1\right) < 0$, that is, for values of $x_{\mathcal{D}}$ satisfying $x_{\mathcal{D}} > 1$.

B. Model predictions under alternative debt limit estimates

We calculate government utility costs and fiscal tipping points relying on the parameters in the legend of Figure 1, and using a default boundary equal to the Natural Public Debt Limit (NPDL) of Section 3.5.2 (see Eq. (9)),

$$\bar{\delta}_{\mathrm{n}} = \frac{\bar{s}}{\bar{\imath}_{\mathrm{n}} - \bar{\mu}}. \tag{I-3}$$

B.1. Benchmark model

Figure I-1 depicts governments utility costs using $\bar{\imath}_{\mathrm{n}} = 0.03$ and $\bar{\mu} = 0.01$, resulting from different assumptions on $\bar{s}$. The top panel relies on the assumption of a "tight debt limit," i.e., resulting from $\bar{s} = 0.01$. The bottom panel is obtained while assuming a "large debt limit," $\bar{s} = 0.03$. In both cases, default occurs in the surplus regime.

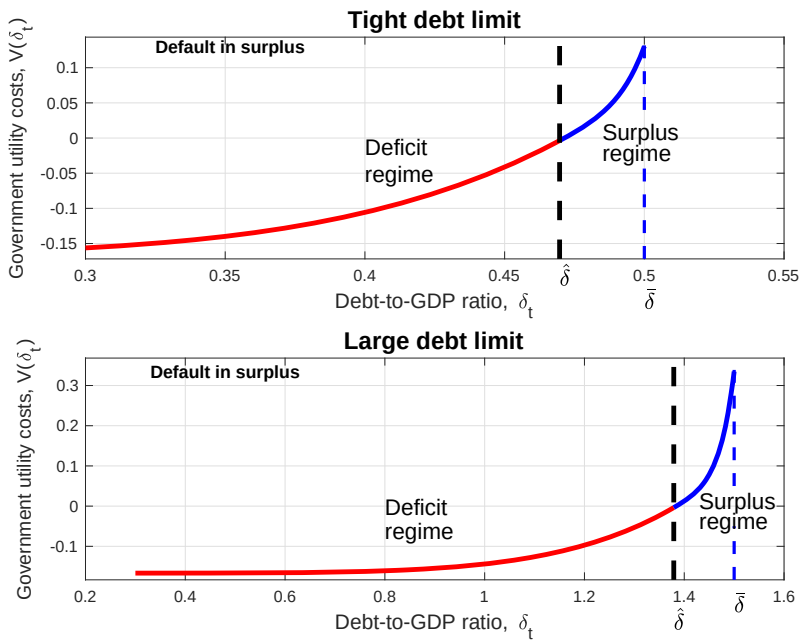


FIGURE I-1: GOVERNMENT UTILITY COSTS. This picture depicts government utility costs resulting from different assumptions on the Natural Public Debt Limit, i.e., δ_{n} in Eq. (I-3), with $\bar{\imath}_{\mathrm{n}} = 0.03$ and $\bar{\mu} = 0.01$. The top panel depicts $V(\delta_t)$ when $\bar{s} = 0.01$ and the bottom panel depicts $V(\delta_t)$ when $\bar{s} = 0.03$. Remaining parameters are as in the legend of Figure 1 in the main text.

Next, we provide results regarding fiscal tipping points in the two cases with tight debt limits (Section B.1) and large debt limits (Section B.2), while setting all remaining parameters equal to those in Figure I-1 and in the variants identified by the next figures.

B.1.1. Tight debt limits

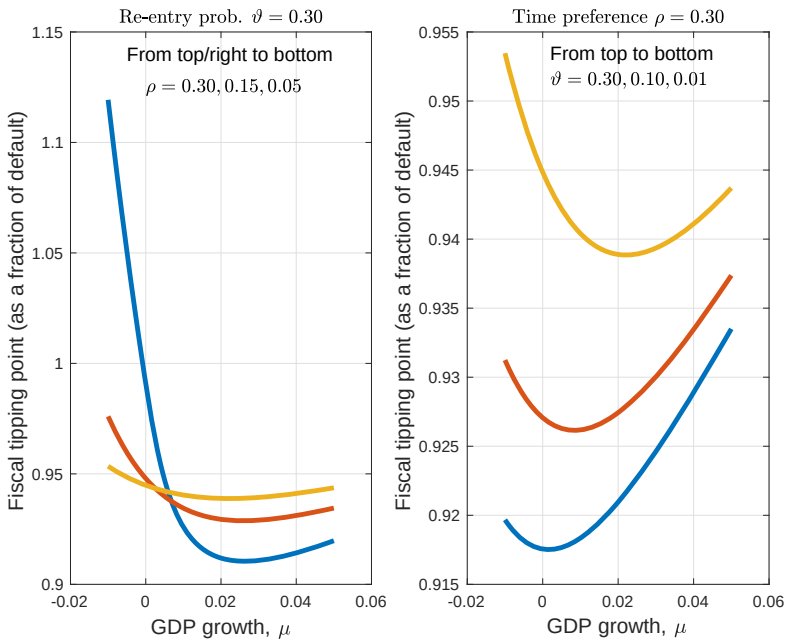


Figure I-2: Fiscal tipping points, government time preferences and probability of re-entry after default.

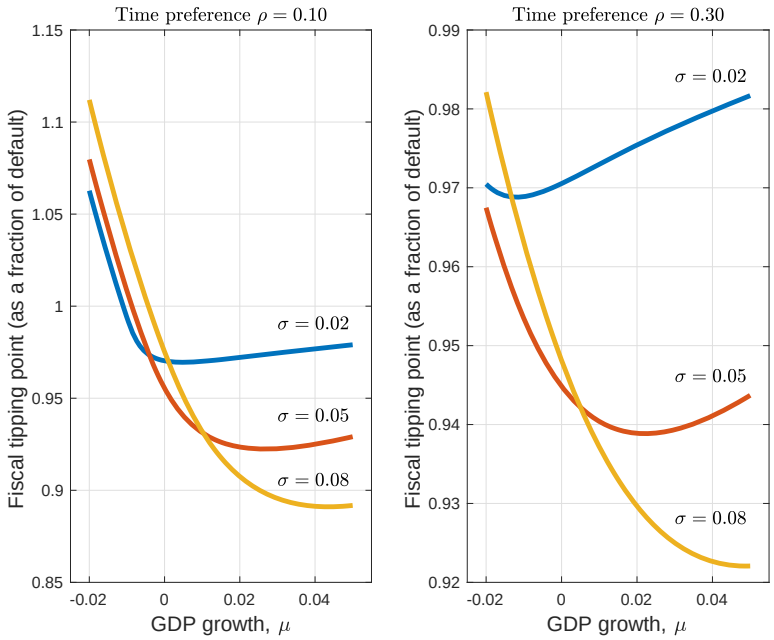


Figure I-3: Fiscal tipping points and macroeconomic volatility.

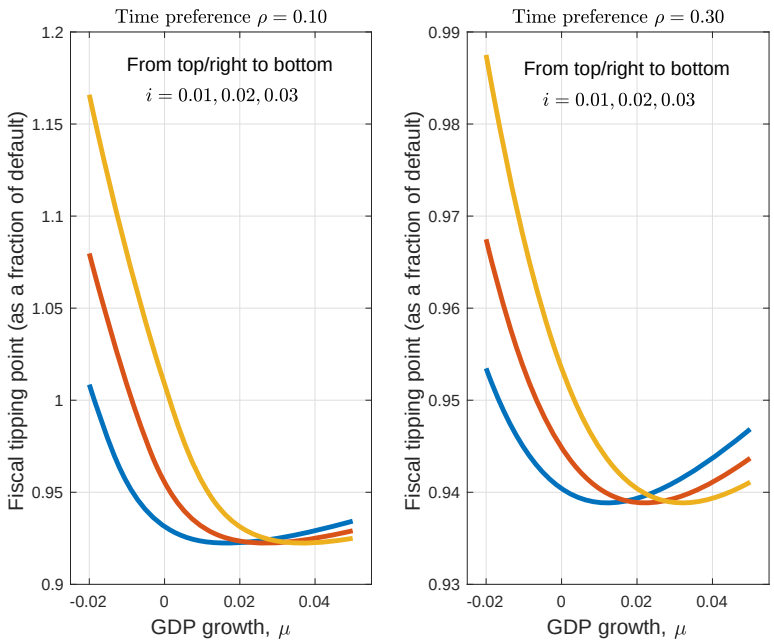


FIGURE I-4: FISCAL TIPPING POINTS AND SHORT-TERM FINANCING COSTS.

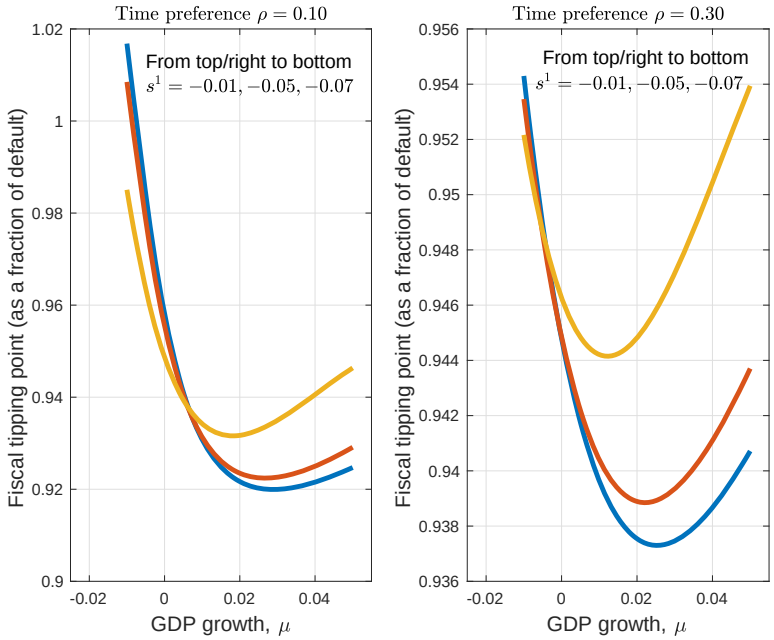


FIGURE I-5: FISCAL TIPPING POINTS AND BUDGET SIZES (DEFICIT).

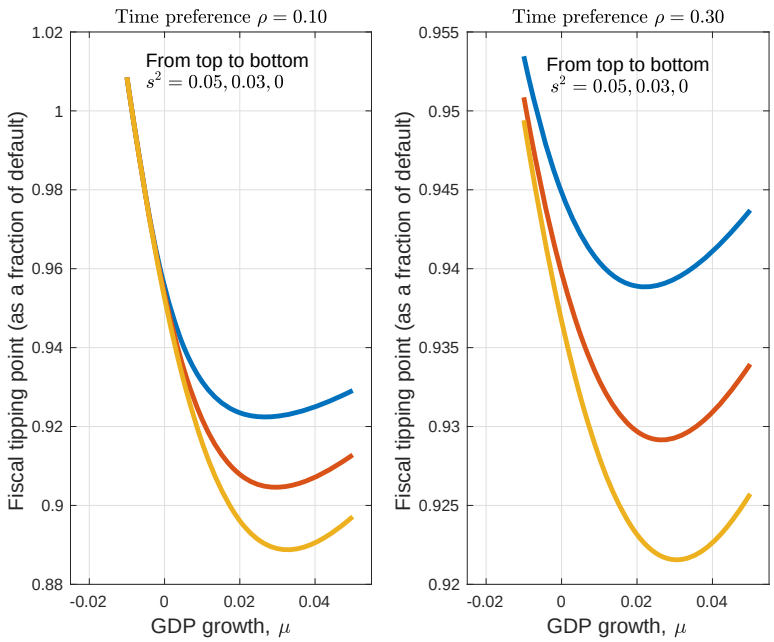


FIGURE I-6: FISCAL TIPPING POINTS AND BUDGET SIZES (SURPLUS).

B.1.2. Large debt limits

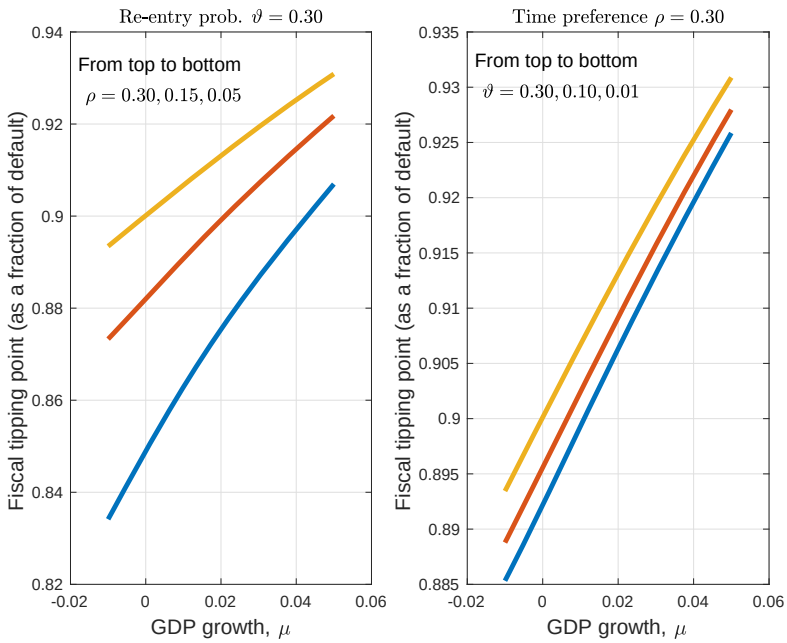


Figure I-7: Fiscal tipping points, government preferences and probability of re-entry after default.

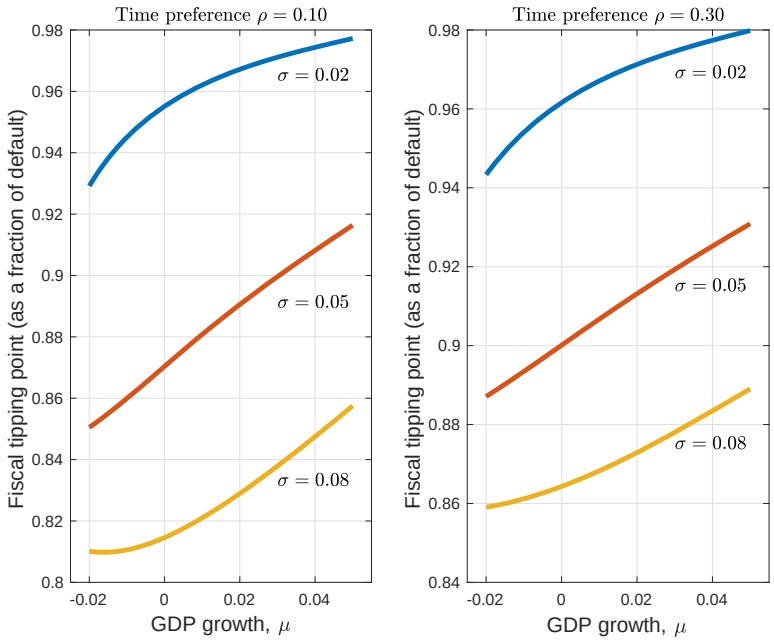


Figure I-8: Fiscal tipping points and macroeconomic volatility.

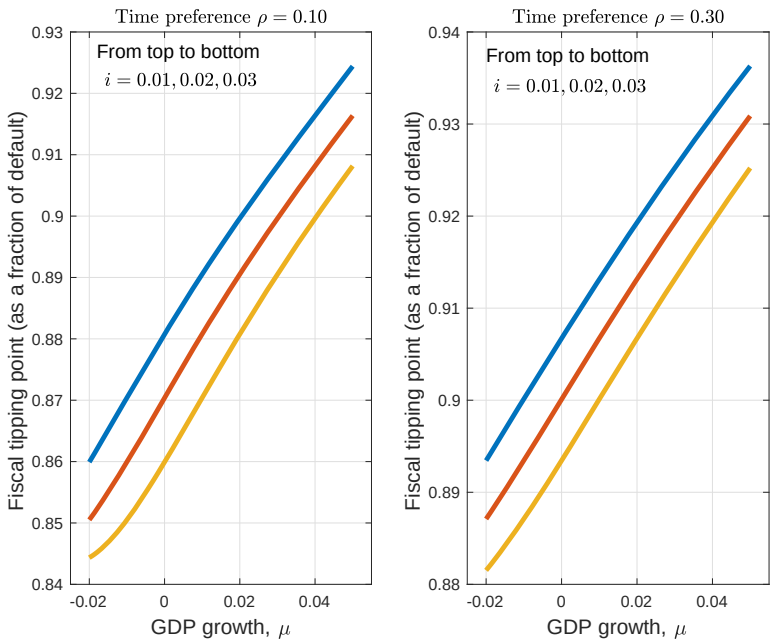


FIGURE I-9: FISCAL TIPPING POINTS AND SHORT-TERM FINANCING COSTS.

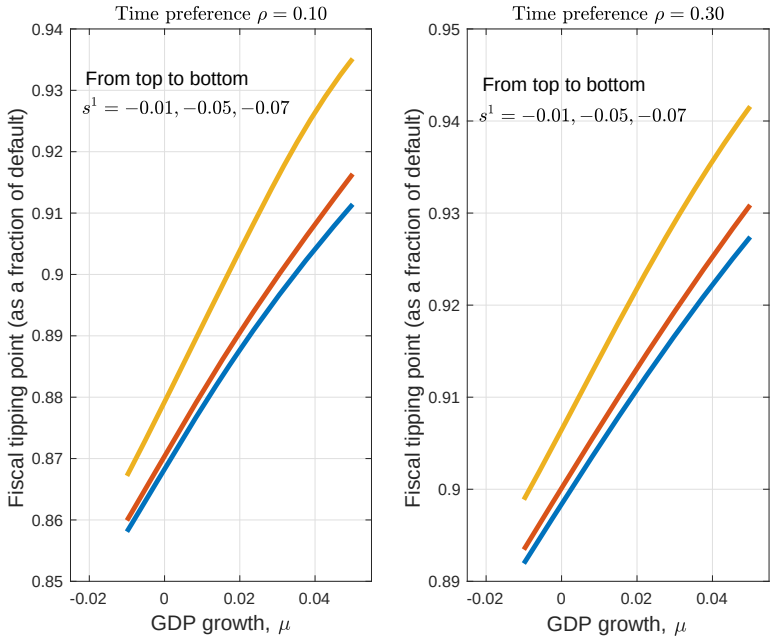


FIGURE I-10: FISCAL TIPPING POINTS AND BUDGET SIZES (DEFICIT).

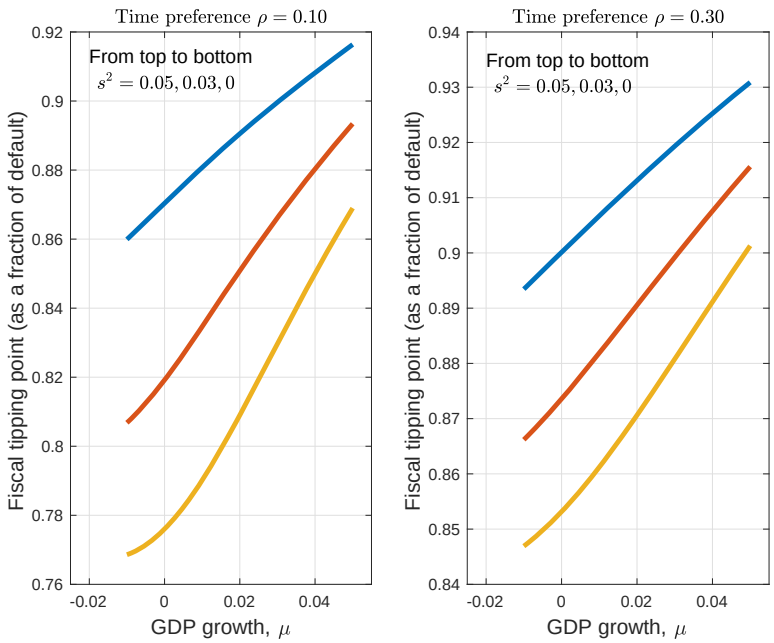


FIGURE I-11: FISCAL TIPPING POINTS AND BUDGET SIZES (SURPLUS).

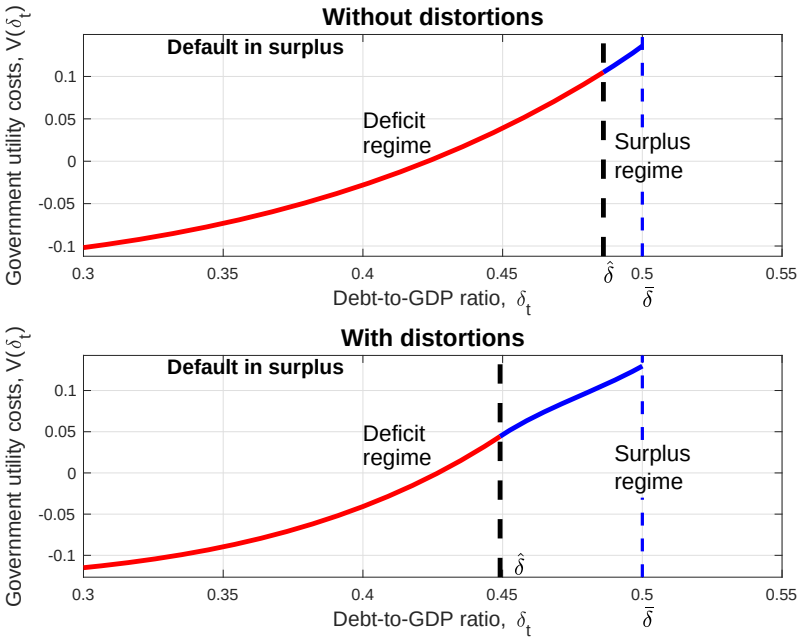
B.2. Fiscal reforms and state-dependent interest rates

We calculate government utility costs, fiscal tipping points and long term spreads relying on the same parameter values in Internet Appendix B.1, with some exceptions explained in a moment.

- Figures I-12 and I-13 in Section B.2.1 depict government utility costs that are the NPDL counterpart of Figures A-3 (Distortionary policies) and A-5 (State-dependent interest rates) in Appendix C of the main paper.
- Sections B.2.2 and B.2.3 provide results on fiscal tipping points, which are the NPDL counterpart (in the tight and large debt limit cases) to those contained in Figures A-4 (Budget sizes and growth) and A-6 (Interest rates) in Appendix C of the main paper. They also provide the NPDL counterpart to results on default premiums summarized by Figures 9 and 10 (on fiscal reforms) and 11 (on the effects of state-dependent interest rates) of the main paper.

B.2.1. Government utility costs

Tight debt limits



Large debt limits

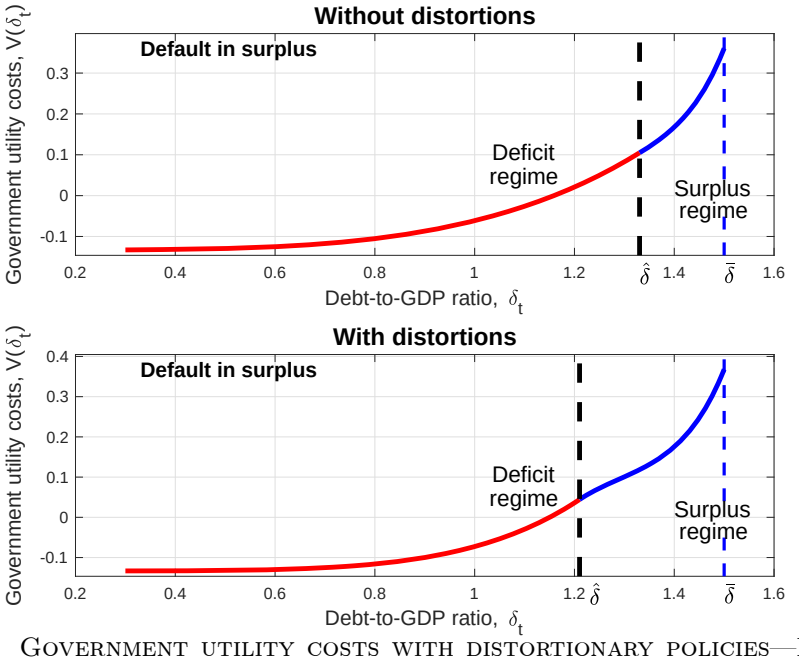
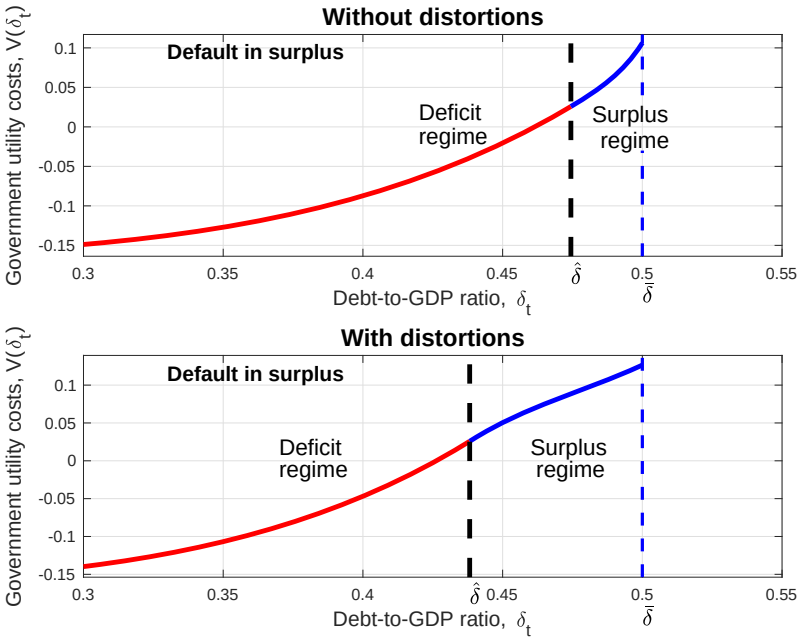


FIGURE I-12: GOVERNMENT UTILITY COSTS WITH DISTORTIONARY POLICIES—BUDGET SIZES AND GROWTH.

Tight debt limits



Large debt limits

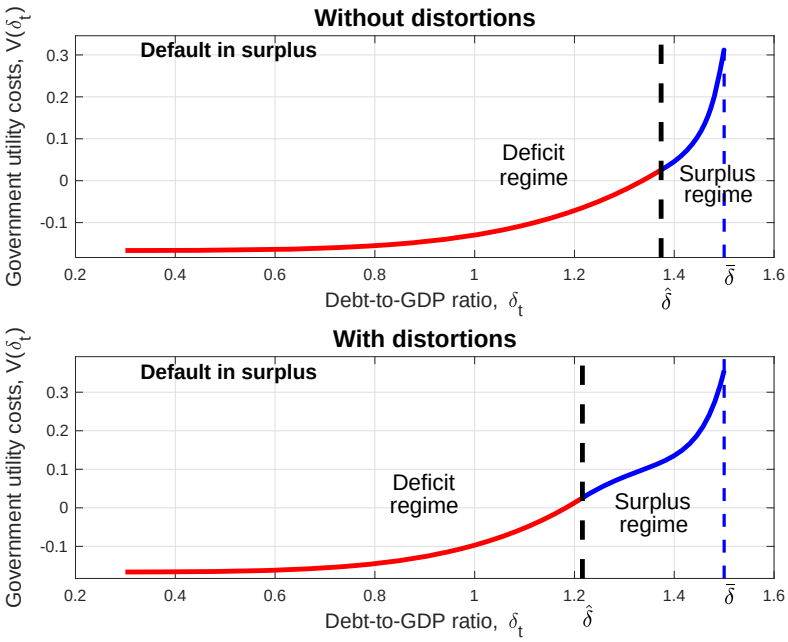


FIGURE I-13: GOVERNMENT UTILITY COSTS WITH DISTORTIONARY POLICIES—INTEREST RATES.

B.2.2. Fiscal tipping points and spreads (tight debt limits)

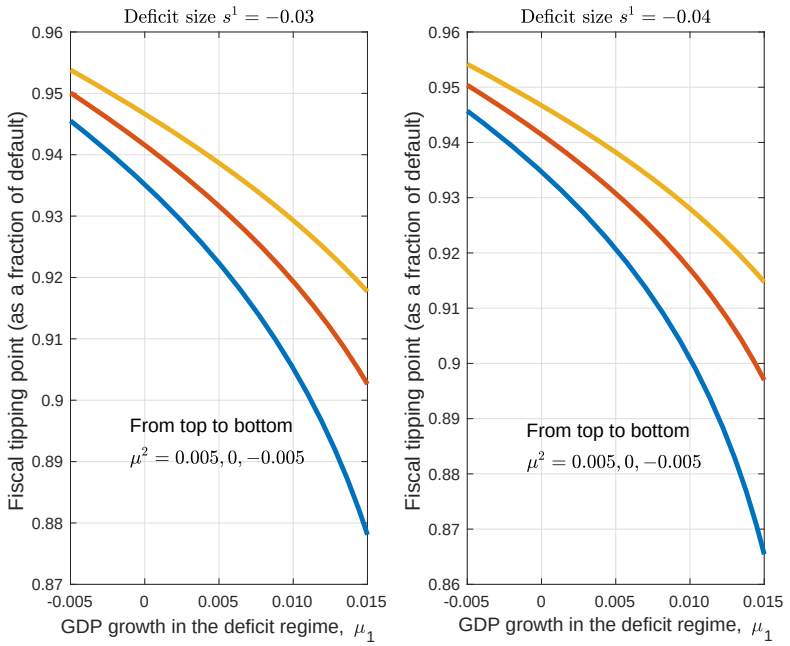


FIGURE I-14: FISCAL TIPPING POINTS AND DISTORTIONARY POLICIES—BUDGET SIZES AND GROWTH.

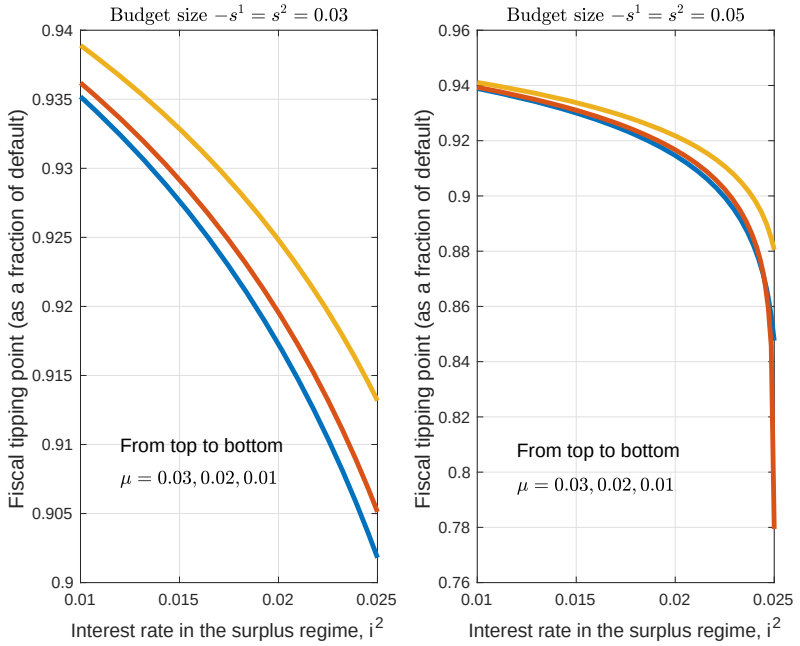


FIGURE I-15: FISCAL TIPPING POINTS AND DISTORTIONARY POLICIES—INTEREST RATES.

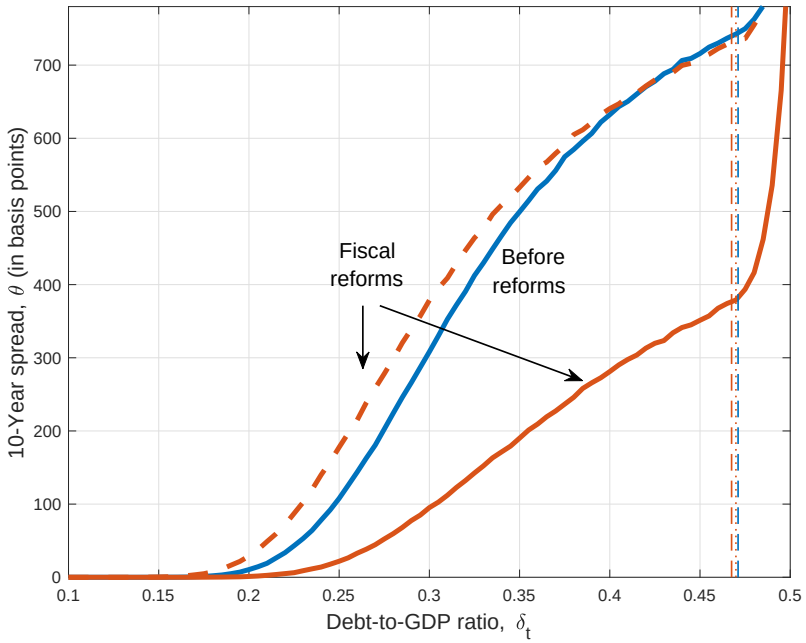


FIGURE I-16: SPREADS AND FISCAL REFORMS, I. This picture depicts fiscal reform experiments that are the (tight) NPDL counterparts to those in Figure 10 of the main text.

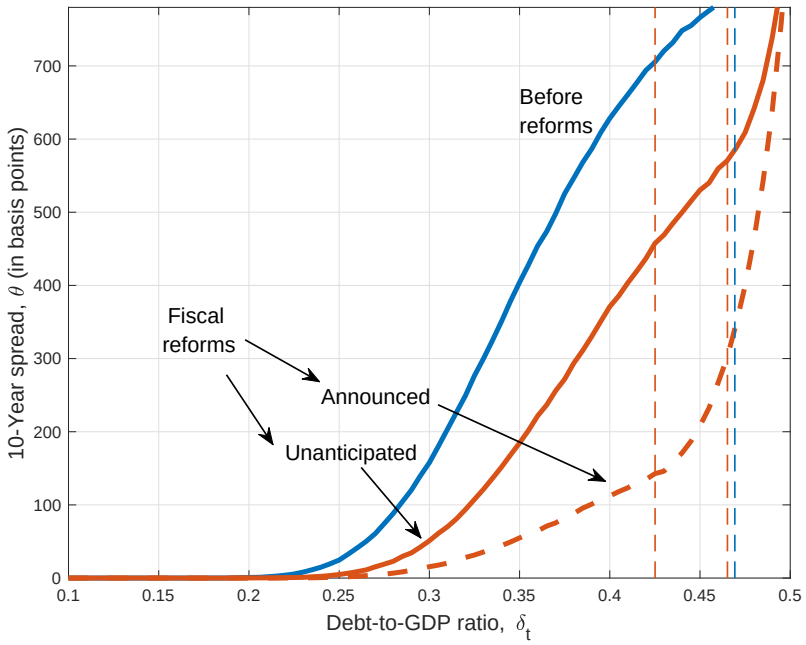


FIGURE I-17: SPREADS AND FISCAL REFORMS, II. This picture depicts fiscal reform experiments that are the (tight) NPDL counterparts to those in Figure 11 of the main text.

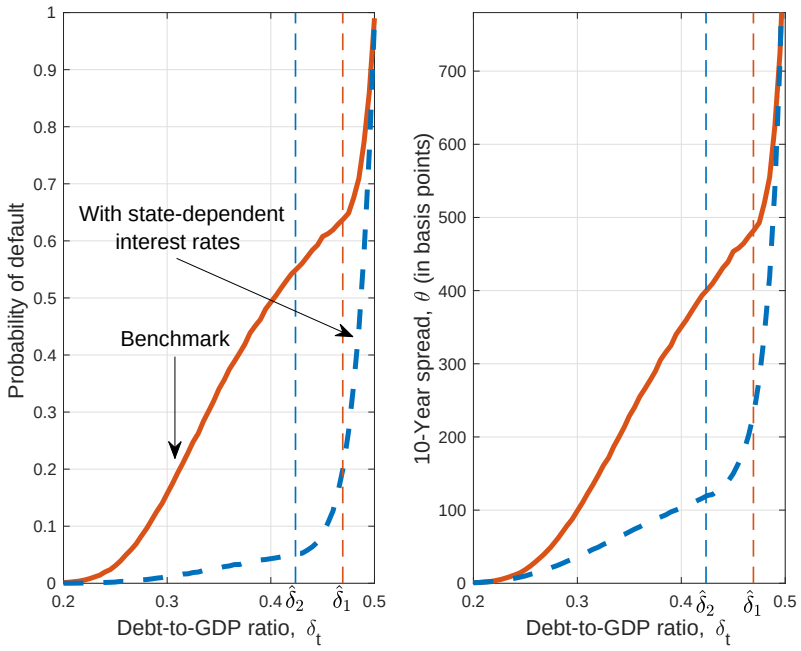


FIGURE I-18: SPREADS AND STATE-DEPENDENT INTEREST RATES.

B.2.3. Fiscal tipping points and spreads (large debt limits)

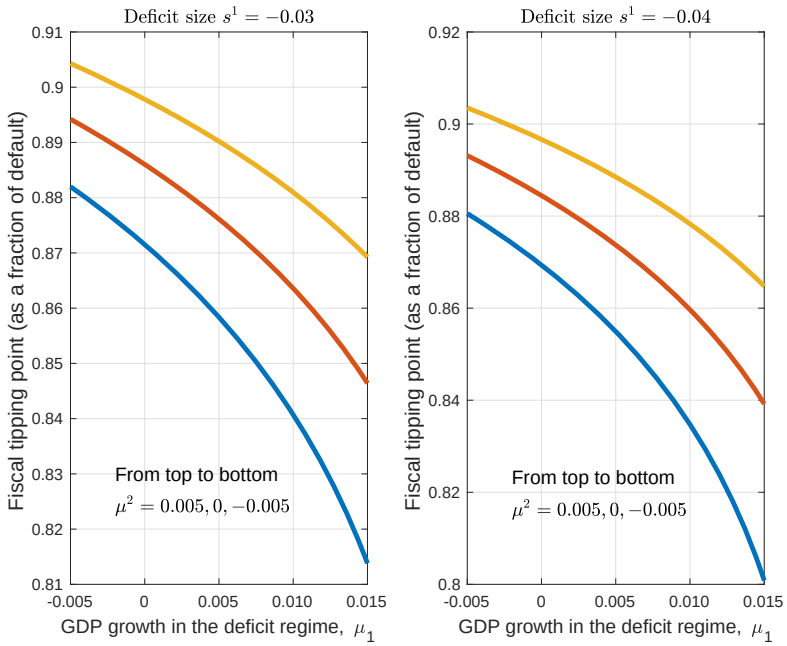


Figure I-19: Fiscal tipping points and distortionary policies—Budget sizes and growth.

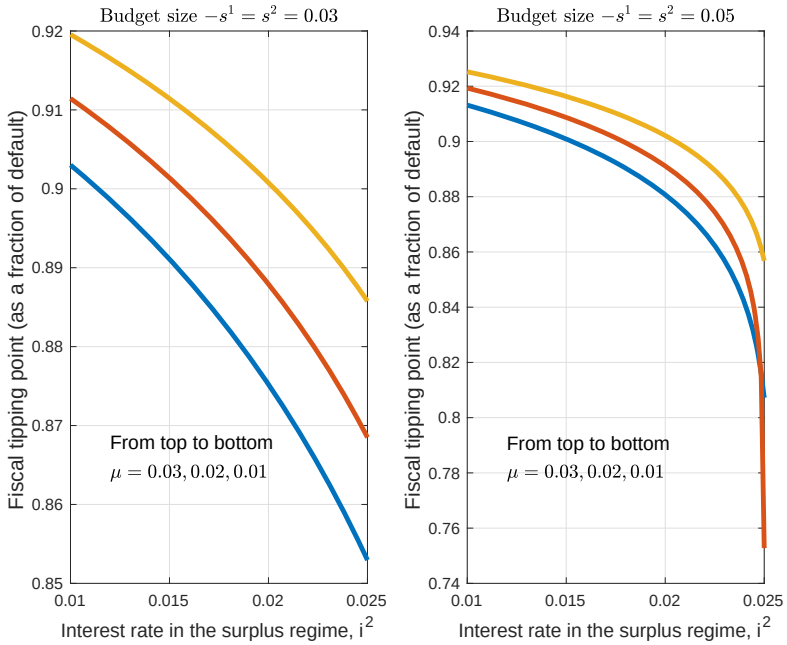


Figure I-20: Fiscal tipping points and distortionary policies—Interest rates.

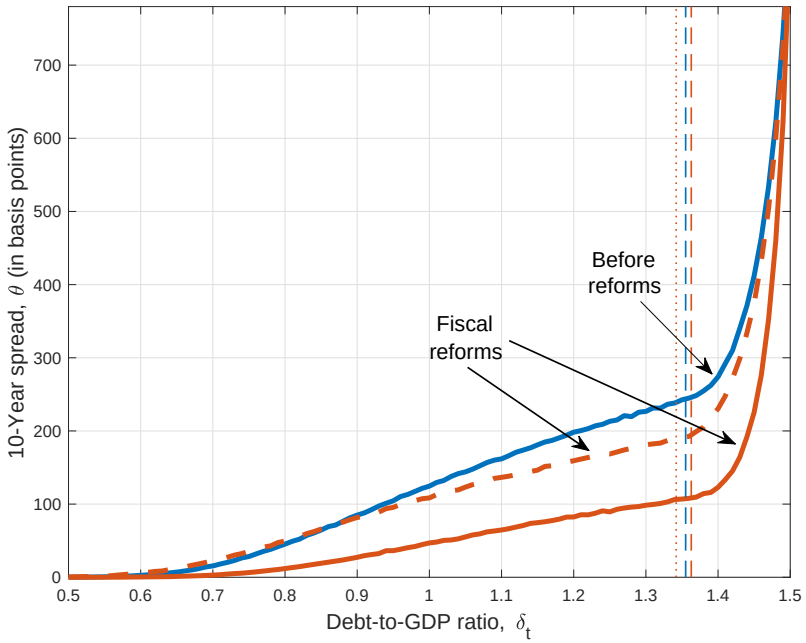


FIGURE I-21: SPREADS AND FISCAL REFORMS, I. This picture depicts fiscal reform experiments that are the (large) NPDL counterparts to those in Figure 10 of the main text.

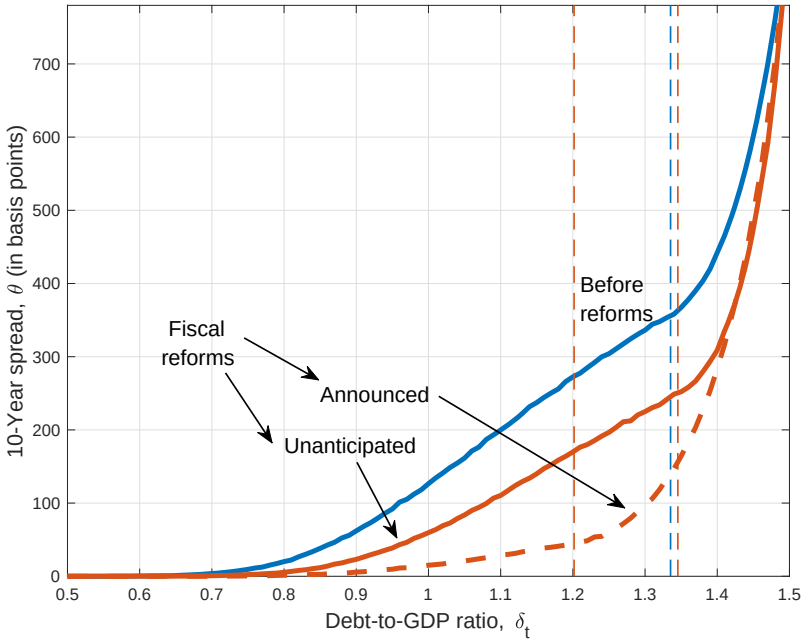


FIGURE I-22: SPREADS AND FISCAL REFORMS, II. This picture depicts fiscal reform experiments that are the (large) NPDL counterparts to those in Figure 11 of the main text.

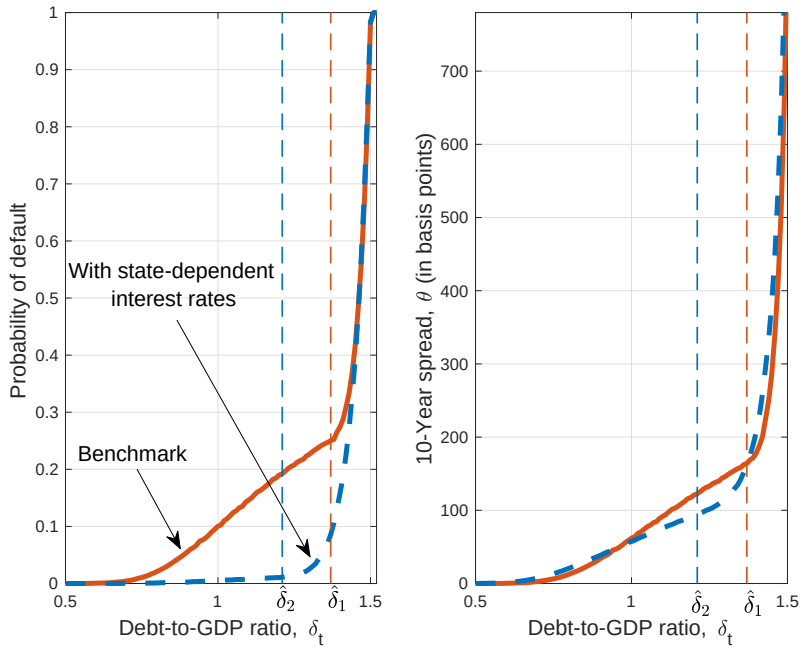


FIGURE I-23: SPREADS AND STATE-DEPENDENT INTEREST RATES.

C. Predictions regarding the model with strategic default

The verification conditions in Appendix C of the main text require that γ be small and additional restrictions on the size of deficits, growth and discount rates (see Lemma C.1 in Appendix C of the main text). They imply that the government utility costs are now *concave* in δ_t (see Figure I-24). Moreover, the governments' policy may be to default in the deficit regime, provided the time preference parameter ρ is small, and may be to default in the surplus regime. The model predicts that the higher ρ, the higher the default boundary. Moreover, for values of ρ sufficiently high, a government would never choose to default. Intuitively, governments have a chance to re-entry financial markets after default; the lower ρ, the more they value this opportunity. These effects are similar to the "gamble-for-resurrection" property in the benchmark model of the main text (see Figure 2 in Section 4). Finally, whether governments default in deficit or in surplus (if any) depends on parameter values, as elaborated in Appendix C.2 and in further experiments below. Interestingly, we find that, provided μ is low enough, default occurs in deficit regimes.

Figure I-24 shows how the probability of re-entry, ϑ, and the costs of financial exclusion, ϵ, affect default boundaries and fiscal tipping points. The first two panels show that an increase in ϑ leads governments to default earlier. In both cases, default occurs whilst governments run deficits, and $\epsilon = 0$; when $\epsilon > 0$ (the third panel), governments default in surplus. That is, governments have increased incentives to default earlier when they face lower costs of financial exclusion. Thus, a higher probability of re-entry may lead to "serial defaulting," the phenomenon by which some countries experience several default episodes over time. This phenomenon, described in the paper of Reinhart, Rogoff and Savastano (2003) mentioned in the main text, links to default occurring at relatively low levels of debt and, then, to "debt intolerance," i.e., the fact that a country may face a high cost of financing even when its debt-to-GDP ratio is relatively small. In our model, a high ϑ lowers $\bar{\delta}_o$, which increases the probability of default; but a higher ϑ also implies that a government is likely to re-entry sooner, with relatively high incentives to default again. Therefore, debt issued by governments with higher ϑ should command a higher premium.

The last two panels show that the higher ϵ, the later governments default, as anticipated. The last panel actually reveals that if the costs of financial exclusion are sufficiently high, governments choose to go through a period of austerity and, eventually, default in the surplus regime. It may be shown that the default boundary also increases with the "haircut" γ. Thus, costs of financial exclusion and haircut are both deterrents for default (contributing, then, to low financing costs). Finally, the default boundary, $\bar{\delta}_o$, now decreases with ξ. When governments control how much time to spend whilst servicing their debt, they will be incentivized to default earlier to avoid large costs at default, which are equal to $\xi\bar{\delta}_o$. Thus, when costs at default are proportional to the stock of existing debt, a high sensitivity of these costs to debt (i.e., a high ξ) can be another channel to early defaulting.

How does uncertainty affect the time to default? Intuitively, this timing is determined by how volatility affects the expected utility costs $E_t\left(V\left(\delta_{t+\Delta t}\right)\right)$ (see Eq. (14) in the main text). These costs are concave in δ_t, but it is possible to show that V is increasing in σ^2, that is, governments default early when facing high macroeconomic uncertainty. This result is due to the property that the drift of δ_t and, thus, $V\left(\delta_t\right)$, increases with σ^2, due to a Jensen's inequality effect (see Eq. (1) in the main text). However, these conclusions are reversed once one accounts for this mechanical effect, such that, then, $V\left(\delta_t\right)$ decreases with σ^2: the option to wait is high when macroeconomic uncertainty is high. In particular, Figures I-25 and I-26 depict default boundaries and fiscal tipping points (as a fraction of the default boundary) as a function of growth, based on two parametric assumptions. The first (in Figure I-25) is that the parameter κ in the drift of the debt-to-GDP ratio is $\kappa = \mu - i - \sigma^2$, as in Eq. (1) of the main text. The second (in Figure I-26) is that $\kappa = \mu - i$. Both pictures illustrate the effects of alternative assumptions on macroeconomic volatility on default boundaries and fiscal tipping points. In Figure I-25, governments default earlier as macroeconomic volatility increases. In Figure I-26, macroeconomic volatility does not affect growth of the debt-to-GDP ratio, such that default boundaries increase and fiscal tipping points decrease with macroeconomic volatility.

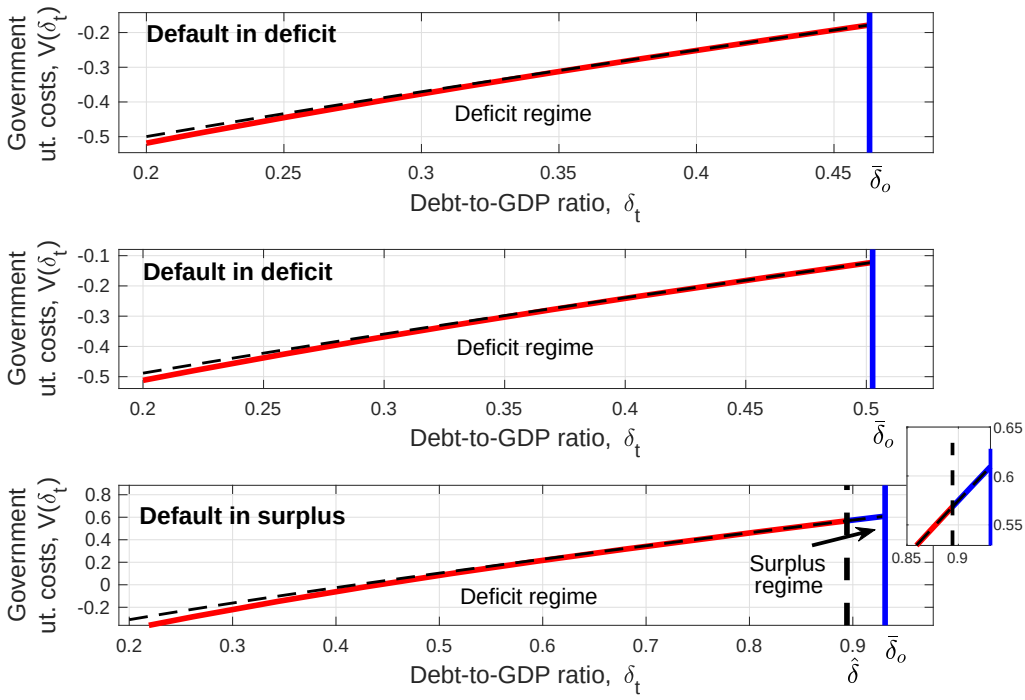


FIGURE I-24: GOVERNMENT UTILITY COSTS IN A MODEL WITH STRATEGIC DEFAULT. This picture depicts government utility costs when the government strategically defaults in a deficit (top and middle panel) and in a surplus (bottom) regime. Austerity costs and probabilities of re-entry after default are $\epsilon = 0$ and $\vartheta = 0.33$ (top panel), $\epsilon = 0$ and $\vartheta = 0.30$ (middle) and, finally, $\epsilon = 0.04$ and $\vartheta = 0.30$ (bottom). Remaining parameters are as follows:

s^1	s^2	ρ	γ	ξ	μ	σ	i	$\bar{\iota}$
-0.05	0.05	0.05	0.50	0.50	0.01	0.15	0.02	0

The dashed lines in all panels are the costs of defaulting. In each panel, $\bar{\delta}$ denotes the default boundary. In the bottom panel, $\hat{\delta}$ is the fiscal tipping point. These paramater values imply that the exogenous default boundary is $\bar{\delta} = 1.4188$ in all cases.

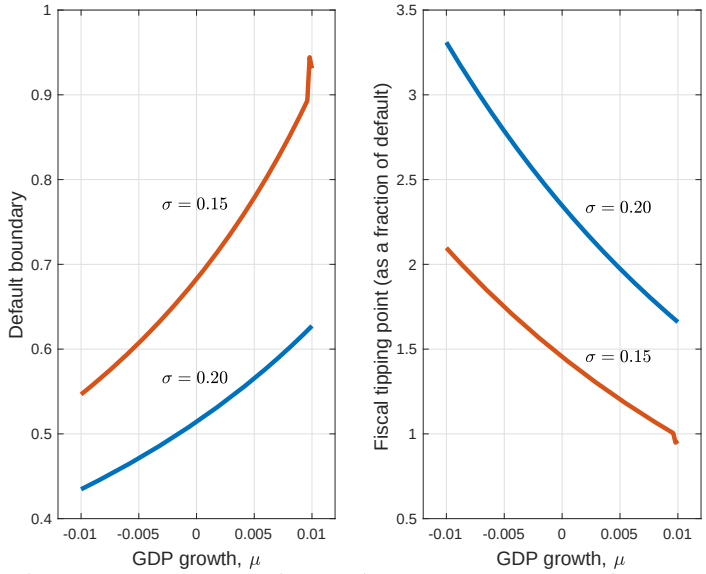


FIGURE I-25: FISCAL TIPPING POINTS AND MACROECONOMIC VOLATILITY (WITH JENSEN'S INEQUALITY EFFECTS). This picture depicts values of the debt-to-GDP ratios that triggers governments' default and a change in the fiscal regime, as a function of growth, μ. The left (resp., right) panel depicts the default boundary $\bar{\delta}_o$ (resp., the fiscal tipping point-to-default boundary, $C = \frac{\hat{\delta}}{\bar{\delta}_o}$) for values of volatility $\sigma = 0.15$ and $\sigma = 0.20$. Remaining parameter values are as in the bottom panel o

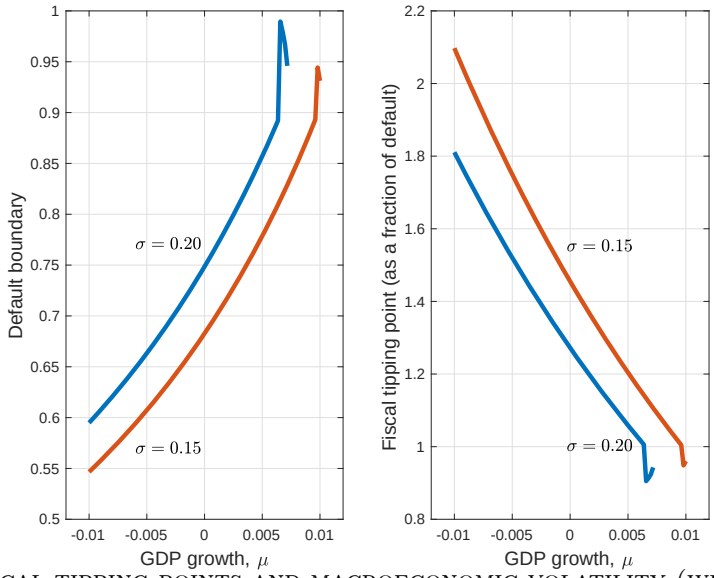


FIGURE I-26: FISCAL TIPPING POINTS AND MACROECONOMIC VOLATILITY (WITHOUT JENSEN'S INEQUALITY EFFECTS). This picture depicts values of the debt-to-GDP ratios that triggers governments' default and a change in the fiscal regime, as a function of growth, μ. The left (resp., right) panel depicts the default boundary $\bar{\delta}_o$ (resp., the fiscal tipping point-to-default boundary, $C = \frac{\hat{\delta}}{\bar{\delta}_o}$) for values of volatility $\sigma = 0.15$ and $\sigma = 0.20$. Remaining parameter values are as in the bottom panel of Figure 2, but with $\mu + \sigma^2$ replacing μ.

D. Primary surplus and business cycles

Surpluses are pro-cyclical (see Footnote 7 of the main text). Define $s_{y,t} \equiv s_t \delta_t$, the surplus-to-GDP ratio. The model predicts that, prior to reaching the fiscal tipping point (when $s_t = s^1$), $s_{y,t}$ increases in absolute value during recessions: deficits increase in bad times. Beyond the tipping point (when $s_t = s^2$), $s_{y,t}$ increases in recessions: now, surpluses increase in bad times. To account for cyclical components affecting primary surpluses, we extend Eq. (1) to include a direct linkage to business cycles.

Consider the government constraint: $dD_t = -d\mathcal{S}_t + iD_t dt$, where $\mathcal{S}_t$ denotes the primary *cumulative* surplus. We assume that the primary surplus, $d\mathcal{S}_t$, is affected by innovations in output, just as in Eq. (3) of the main text, such that the debt-to-GDP ratio δ_t is still a Markov process

$$d\delta_t = -\left(s_t + \kappa\right)\delta_t dt - \sigma\left(1+\psi\right)\delta_t dW_t. \tag{I-4}$$

Now, the surplus-to-debt ratio is $d\mathcal{S}_t^D = \frac{d\mathcal{S}_t}{D_t}$, and the government program is

$$V\left(\delta_t\right) = \inf_{s_u \in [s^1, s^2]} E_t\left[\int_t^\infty e^{-\rho(u-t)} d\mathcal{S}_t^D\right],$$

under the constraint that the debt-to-GDP ratio is solution to (I-4). Solutions to government plans are just as those in the main text, but with volatility parameter $\sigma\left(1+\psi\right)$ replacing σ.

Surplus-to-GDP ratios are, now, $d\mathcal{S}_t^y = \delta_t d\mathcal{S}_t^D$, and may well be pro-cyclical, as in the following simulations. We simulate δ_t in (I-4) and calculate the primary surplus $d\mathcal{S}_t^y$ and output growth. We perform 1000 simulations of 50 years of data, and calculate an average primary surplus and an average output growth for each year, conditioned on no defaults having taken place. We set the cyclical component parameter equal to $\psi = 1$, and keep remaining parameter values are those in the legend of Figure 1 of the main text. Figure I-27 depicts a scatterplot of the primary surplus against output in these simulations.

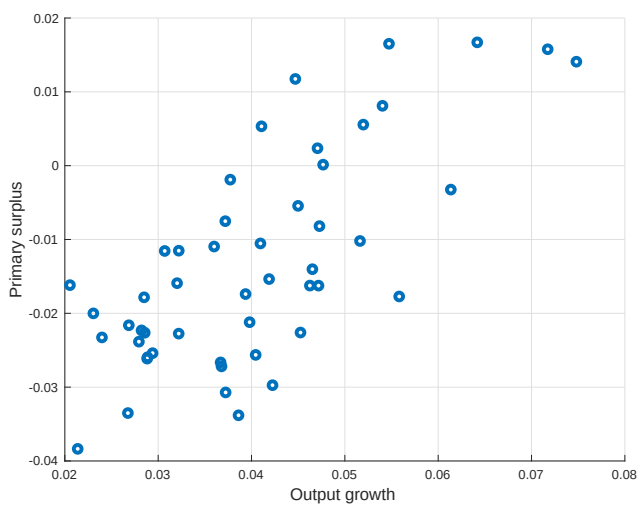


Figure I-27: Simulated primary surpluses and business cycles.